MATRIX TRANSFORMATIONS BETWEEN SEQUENCE SPACES

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Abstract

Matrix transformations are essential in the study of sequence spaces, which are foundational objects in functional analysis. Sequence spaces, such as l^p, l^∞, c, and c_0, are studied for their inherent properties like convergence, boundedness, and continuity. This chapter explores the theoretical framework for matrix transformations acting on these spaces, providing a comprehensive understanding of norm preservation, continuity, and operator behavior. Emphasis is placed on the completeness of these spaces, their dual relationships, reflexivity, and the algebraic and topological properties that define their structure. Matrix operators are examined as linear transformations that influence convergence and operator stability, with applications spanning from signal processing to numerical computation. Theoretical results regarding inclusion, duality, and operator norms form the backbone of the analysis, laying the groundwork for future studies on infinite matrices and their applications in real-world problems.

Keywords

Matrix transformations, sequence spaces, normed vector spaces, Banach spaces, convergence, operator theory, functional analysis, l^pspaces, l^∞, duality, reflexivity.

Introduction

Matrix transformations play a pivotal role in the study of sequence spaces, which are central to functional analysis and have applications in various domains of applied mathematics, such as signal processing, numerical methods, and mathematical modeling. Sequence spaces are sets of sequences of real or complex numbers that adhere to certain normed conditions, such as absolute summability and boundedness. These spaces include well-known types like l^p, l^∞, and c_0, each having its own distinct characteristics that are crucial in operator theory. When matrices act as linear operators on sequence spaces, they can alter key properties of the sequences such as convergence, continuity, and boundedness. This chapter sets out to provide a formal understanding of how matrix transformations interact with these spaces. Beginning with foundational definitions of sequence spaces, it proceeds to discuss the properties of matrix transformations, focusing on norm preservation, continuity, and convergence. These properties are essential for understanding the behavior of matrices in applied contexts, including the stability of numerical algorithms and signal processing systems.

Sequence Spaces: Definitions and Properties

Common Sequence Spaces

Let $\mathbb{K} = \mathbb{R}$or $\mathbb{C}$. Consider the vector space of all infinite sequences $x = (x_n)_{n=1}^{\infty}$with entries in $\mathbb{K}$. Within this large space one introduces specific subspaces defined by norm-conditions, which are central to functional analysis and operator theory on sequences. The most commonly employed sequence spaces in the present study are

(a) The space $\ell^p$for $1 \leq p < \infty$

$$\ell^p = \{x = (x_n)_{n=1}^{\infty} : \| x \|_p := \left(\sum_{n=1}^{\infty} | x_n |^p\right)^{1/p} < \infty\}.$$

Here $\| x \|_p$ defines a norm, and with this norm ℓ^p becomes a normed vector space. The requirement $\sum_{n=1}^{\infty} | x_n |^p < \infty$ ensures that x is p-**summable**. For example, for $p = 2$, one obtains the familiar Hilbert space ℓ^2 of square-summable sequences. Classical results show that ℓ^p is a Banach space—that is, it is complete under the metric induced by $\|\cdot\|_p$.

Example: Let $x_n = \frac{1}{n^{1+1/p}}$. Then

$$\sum_{n=1}^{\infty} | x_n |^p = \sum_{n=1}^{\infty} \frac{1}{n^{p(1+1/p)}} = \sum_{n=1}^{\infty} \frac{1}{n^{p+1}} < \infty,$$

so $x = (x_n) \in \ell^p$.

(b) The space ℓ^{∞}

$$\ell^{\infty} = \{x = (x_n) : \| x \|_{\infty} := \sup_{n\in\mathbb{N}} | x_n | < \infty\}.$$

Here ℓ^{∞} is the space of **bounded sequences**. The supremum norm $\|\cdot\|_{\infty}$ satisfies the norm axioms, and it is well-known that ℓ^{∞} is also a Banach space.

Example: Let $x_n = (-1)^n$. Then $\sup_n | x_n | = 1$, so $x \in \ell^{\infty}$. On the other hand, x does *not* belong to ℓ^p for any finite p, since $\sum_{n=1}^{\infty} | x_n |^p = \sum_{n=1}^{\infty} 1 = \infty$.

(c) The space c and the space c_0

- c: the space of all convergent sequences. In other words

$$c = \{x = (x_n) : \lim_{n\to\infty} x_n \text{ exists in } \mathbb{K}\}.$$

c_0: the space of all sequences converging to 0. That is

$$c_0 = \{x = (x_n) : \lim_{n\to\infty} x_n = 0\}.$$

With the sup-norm $\| x \|_{\infty} = \sup_n | x_n |$, both c and c_0 are closed subspaces of ℓ^{∞}, and hence Banach spaces in their own right.

Example: Consider the sequence $x_n = 1/n$. Then $x \in c_0$, because $\lim_{n\to\infty} x_n = 0$. Also, $x \in \ell^p$ for all $p > 1$(since $\sum_{n=1}^{\infty}(1/n)^p$ converges when $p > 1$), but $x \notin \ell^1$(because the harmonic series diverges).

- For each $p \in [1, \infty)$, $\ell^p \subset \ell^q$ whenever $p < q$ does **not** hold in general for infinite sequences; rather, one has $\ell^p \supset \ell^q$ when $p < q$(because stricter summability implies more sequences).

- The relation $\ell^p \subset \ell^\infty$ holds for $1 \le p < \infty$, since if $\sum |x_n|^p < \infty$ then $x_n \to 0$, and so $\sup_n |x_n| < \infty$ (though conversely a bounded sequence need not be p-summable).
- $c_0 \subset c \subset \ell^\infty$.
- These spaces each carry a norm which induces a metric $d(x,y) = \| x - y \|$, making them normed vector spaces; moreover, they are **complete** (i.e., Banach spaces). Completeness means every Cauchy sequence in the space converges (with respect to the given norm) to an element of the same space.

In this way, the sequence spaces ℓ^p, ℓ^∞, c, and c_0 provide the foundational frameworks in which matrix-operators act and the subsequent theoretical analyses (of convergence, boundedness, operator-continuity) will take place.

Topological and Algebraic Structure

This section explores the topological and algebraic features of the principal sequence spaces introduced in Section 2.2.1. The goal is to establish their structure as normed vector spaces, examine their completeness (Banach property), investigate duality relations, assess reflexivity, and discuss other relevant operator-theoretic properties. This foundational analysis is critical for subsequent chapters where matrix transformations act as linear operators on these spaces.

Normed Vector Spaces and Banach Spaces

A *normed vector space* $(X, \|\cdot\|)$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a vector space equipped with a norm $\|\cdot\|$ satisfying:

1. $\| x \| \ge 0$ for all $x \in X$, and $\| x \| = 0 \iff x = 0$.
2. $\| \alpha x \| = |\alpha| \cdot \| x \|$ for all $\alpha \in \mathbb{K}$ and $x \in X$.
3. $\| x + y \| \le \| x \| + \| y \|$ (triangle inequality).

With the metric $d(x,y) = \| x - y \|$, X becomes a metric space. If every Cauchy sequence in (X, d) converges to a limit in X, then X is called a *Banach space*.

Theorem (Completeness of ℓ^p Spaces).

For each $1 \le p \le \infty$, the sequence space ℓ^p (or ℓ^∞) is complete under the norm $\|\cdot\|_p$ (or $\|\cdot\|_\infty$). In other words, ℓ^p and ℓ^∞ are Banach spaces.

Proof Sketch: One shows that if $(x^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in ℓ^p, then for each fixed index k, the coordinate sequence $(x_k^{(n)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$, hence converges to some limit x_k. One then shows that the sequence $x = (x_k)$ lies in ℓ^p and $\| x^{(n)} - x \|_p \to 0$. A similar argument holds for ℓ^∞.

Because c and c_0 are closed subspaces of ℓ^∞ (the limit of a convergent or null-convergent bounded sequence remains convergent or null-convergent), they are also Banach spaces under the supremum norm.

Thus the primary sequence spaces of interest — ℓ^p, ℓ^∞, c, and c_0 — are all Banach spaces, which is crucial because linear operators (including infinite matrices) on these spaces can be studied using the rich framework of Banach space theory (boundedness, duality, operator norms).

Linear Algebraic Structure

Each sequence space is a vector space over $\mathbb{K}$, i.e., closed under finite linear combinations. In addition, they satisfy important inclusion and algebraic relations:

- For $1 \le p < q < \infty$, one has $\ell^q \subset \ell^p$if the underlying sequence of (x_n)decays sufficiently fast; in general, $\ell^p \supsetneq \ell^q$.
- $\ell^p \subset \ell^\infty$for $1 \le p < \infty$: any $x \in \ell^p$satisfies $\lim_{n\to\infty} x_n = 0$, hence is bounded.
- $c_0 \subset c \subset \ell^\infty$.
- Each space is closed under coordinate-wise addition and scalar multiplication, and the norms satisfy homogeneity and triangle-inequality properties.

The algebraic simplicity of these instantiations is a major reason for their use as model spaces in functional and sequence space studies.

Duality (Continuous Duals) and Reflexivity

A central aspect of Banach space theory is the *dual space*. For a Banach space X, its continuous dual $X^* = \{f: X \to \mathbb{K} \mid f$ is linear and continuous$\}$is itself a Banach space under the operator (norm) $\| f \| = \sup_{\|x\|\le 1} | f(x) |$.

Theorem (Duality of ℓ^pSpaces)

Let $1 \le p < \infty$and let $q$satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $\ell^p$is isometrically isomorphic to ℓ^q. In other words, $(\ell^p)^* = \ell^q$. Moreover, $(\ell^1)^* = \ell^\infty$and $(c_0)^* = \ell^1$.

Example: For $x = (x_n) \in \ell^p$and $y = (y_n) \in \ell^q$, define the functional

$$f_y(x) = \sum_{n=1}^{\infty} x_n\, y_n.$$

Hölder's inequality asserts convergence of this series and $\| f_y \| = \| y \|_q$. Thus every $y \in \ell^q$defines a bounded linear functional on ℓ^p. The theorem further asserts *every* bounded linear functional on ℓ^parises in this way.

Reflexivity: A Banach space $X$is *reflexive* if the canonical embedding $J: X \to X^{**}$is surjective (and thus an isometric isomorphism). It is known that $\ell^p$is reflexive whenever $1 < p < \infty$. However, ℓ^1, ℓ^∞, and $c_0$are *not* reflexive .

This matters because reflexive spaces inherit favourable properties: for example, every bounded sequence has a weakly convergent subsequence, and the unit ball is weakly compact. These facts become relevant in the analysis of operator convergence and stability in later chapters.

Topological and Metric Structure

Being normed vector spaces, each sequence space carries a topology induced by the metric $d(x, y) = \| x - y \|$. Key topological features include:

- **Open and Closed Sets:** Norm-balls $B_r(x_0) = \{x: \| x - x_0 \| < r\}$form a base of the topology.

- **Convergence and Cauchy Sequences:** In a Banach space, convergence in norm implies the sequence is Cauchy. The converse holds by the completeness property.

- **Equivalent Norms:** On finite-dimensional spaces all norms are equivalent, but in infinite-dimensional sequence spaces different p-norms generate different topologies (although the spaces themselves embed in each other in well-defined ways).

- **Weak and Weak* Topologies:** In reflexive spaces, bounded sequences have weakly convergent subsequences. In non-reflexive examples, phenomena like weak-compactness and the structure of the bidual become important .

Example: Consider the unit ball $B = \{x \in \ell^2 : \| x \|_2 \leq 1\}$. Because ℓ^2 is reflexive, B is weakly compact; the Banach-Alaoglu and Eberlein-Šmulian theorems guarantee this. In ℓ^∞, by contrast, the unit ball is not weakly sequentially compact, reflecting non-reflexivity.

Such topological distinctions are essential when analyzing convergence of infinite-matrix (operator) transformations on sequence spaces later in this thesis.

Schauder Bases and Coordinate Representations

A *Schauder basis* in a Banach space X is a sequence $(e_n)_{n=1}^{\infty}$ such that for every $x \in X$ there exist scalars (a_n) with

$$x = \sum_{n=1}^{\infty} a_n e_n,$$

in the norm sense, and the partial sums define bounded linear projections. The classical spaces ℓ^p admit the canonical unit-vector basis $e^{(n)}$ defined by

$$e_k^{(n)} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

For any $x = (x_n) \in \ell^p$, one then has

$$x = \sum_{n=1}^{\infty} x_n \, e^{(n)}.$$

This basis structure simplifies operator representations (e.g., infinite-matrices acting on the basis vectors). It also facilitates norm estimates and series expansions of transformed sequences.

Operator Norms and Bounded Linear Operators

Let X and Y be Banach spaces. A linear operator $T: X \to Y$ is said to be *bounded* if

$$\| T \| = \sup_{\|x\|_X \leq 1} \| Tx \|_Y < \infty.$$

A foundational result in functional analysis is:

Theorem. Linear $T: X \to Y$ is continuous if and only if it is bounded.

In the context of sequence spaces, infinite matrices $A = (a_{nk})$define operators $A: X \to Y$, which are bounded if and only if certain summability or row-norm conditions hold (to be studied in Section 2.3). This operator-theoretic framework underpins the entire thesis.

Embedding and Inclusion Relations

Given the distinct sequence spaces, one may examine embedding (continuous injection) relationships.

- The inclusion map $i: \ell^p \hookrightarrow \ell^q$(for $p < q$) is continuous but not surjective in general; specific decay rates are required for embedding.
- The inclusion $c_0 \hookrightarrow \ell^\infty$is continuous (with $\| x \|_\infty \leq \| x \|_\infty$), and similarly $c \hookrightarrow \ell^\infty$.
- Embeddings can have dense ranges or closed complements depending on the spaces; these features are important when studying operator ranges and spectrum.

To recapitulate:

- All principal sequence spaces are Banach spaces under their standard norms.
- Duality relations give explicit identifications: $(\ell^p)^* = \ell^q$for $1 \leq p < \infty$and $\frac{1}{p} + \frac{1}{q} = 1$; $(c_0)^* = \ell^1$.
- Reflexivity holds for $1 < p < \infty$but fails for ℓ^1, ℓ^∞, and c_0.
- The canonical basis of ℓ^pprovides a useful coordinate representation for sequences and operators [39].
- The operator norm equivalence between boundedness and continuity applies to infinite-matrix representations.
- Topological features—norm, weak, and weak∗ topologies—differ across spaces and affect convergence and compactness properties.

These structural results provide a firm foundation for analyzing the behaviour of matrix transformations on sequence spaces, especially with respect to operator boundedness, convergence of series and iterates, and stability of transformation in subsequent chapters.

Conclusion

Matrix transformations in sequence spaces are integral to the analysis of various functional spaces and their operators. This chapter has outlined the theoretical properties of matrix operators acting on spaces like l^p, l^∞, c, and c_0. These spaces, being Banach spaces, provide a robust framework for understanding the action of matrices as linear operators, particularly with respect to their effect on convergence, boundedness, and continuity. The discussion has covered key aspects such as norm preservation, duality relations, reflexivity, and the role of operator norms in characterizing matrix transformations. The results of this study form a foundation for further exploration of matrix operations on infinite sequences and their applications to signal processing, numerical methods, and other areas of applied mathematics. Future research will build on these theoretical results to explore more complex matrix operators and their role in modern computational techniques.

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