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GENERALIZED FIXED POINT THEOREMS IN B-METRIC AND COMPLEX-VALUED B-METRIC SPACES

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Abstract

Fixed point theory is a central tool for proving the existence and uniqueness of solutions of nonlinear equations, integral equations, dynamic processes and operator systems. Classical metric fixed point results are powerful, but many contemporary problems require distance structures that are more flexible than ordinary metric spaces. B-metric spaces relax the triangle inequality through a coefficient s >= 1, while complex-valued b-metric spaces extend the range of the distance function from non-negative real numbers to ordered complex numbers. This paper provides a conceptual and analytical study of generalized fixed point theorems in b-metric and complex-valued b-metric spaces. It explains the basic structure of these spaces, reviews major developments from 2009 to 2025, compares contraction principles, and interprets how rational, weak, multivalued and common fixed point conditions operate in generalized metric-type settings. The study adopts a qualitative doctrinal-mathematical method based on secondary scholarly literature and theoretical synthesis. Tables, diagrams and convergence graphs are used to clarify the role of the b-metric coefficient, Picard iteration and complex-valued ordering. The paper argues that the main contribution of b-metric and complex-valued b-metric theory lies not only in formal generalization but also in its ability to handle nonlinear dependence, multicomponent distances and more expressive contractive structures. The findings suggest that future research should focus on sharper coefficient conditions, applications to functional and integral equations, and careful verification of completeness and comparability assumptions.

Keywords: fixed point theorem; b-metric space; complex-valued b-metric space; generalized contraction; common fixed point; rational contraction; Picard iteration; nonlinear analysis

1. Introduction

A fixed point of a mapping T is a point x such that Tx = x. Although this definition is simple, fixed point theory has become one of the most important methods for proving the solvability of mathematical problems. In nonlinear analysis, an equation can often be rewritten as Tx = x, and the existence of a solution is then reduced to the existence of a fixed point. The classical contraction principle works in complete metric spaces, but many mathematical models are not naturally described by an ordinary metric. The distance may be weighted, vector-like, dependent on a coefficient, or connected to complex-valued quantities. For this reason, generalized metric structures have attracted substantial attention in recent fixed point research.

B-metric spaces occupy a significant position in this development because they keep the essential idea of distance but relax the strict triangle inequality. Instead of requiring d(x,z) <= d(x,y) + d(y,z), a b-metric allows d(x,z) <= s[d(x,y) + d(y,z)] for some coefficient s >= 1. This coefficient changes the way Cauchy sequences, convergence estimates and contraction constants are interpreted. Later work extended these ideas to ordered, partial, quasi-partial, vector-valued, extended and complex-valued versions. Among these, complex-valued b-metric spaces are especially interesting because they combine two forms of generalization: the b-metric coefficient and a complex-valued distance ordered through real and imaginary parts.

Recent studies show that fixed point theorems in b-metric spaces are not merely repetitions of classical metric results. The coefficient s can modify the admissible range of contraction constants, while generalized contractive expressions may involve rational terms, control functions, weak comparison functions, multivalued maps or pairs of weakly compatible mappings. Aydi et al. (2012) extended set-valued quasi-contractions in b-metric spaces, Aghajani et al. (2014) studied generalized weak contractions in partially ordered b-metric spaces, and Saadi and Hamaizia (2023) developed multivalued common fixed point results in complex b-metric spaces. These directions indicate that the field is expanding toward more flexible analytical frameworks.

The present paper examines generalized fixed point theorems in b-metric and complex-valued b-metric spaces from

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a conceptual and analytical perspective. It does not attempt to prove a single new theorem. Rather, it organizes the existing theory, explains the conditions under which fixed point conclusions become valid, and identifies the structural differences between real-valued and complex-valued generalized metric settings. The paper is therefore useful for students and researchers who need a coherent framework before moving into technical proof construction.

1.1 Problem Statement

The main problem addressed in this paper is that generalized fixed point results in b-metric and complex-valued b-metric spaces are often presented in highly technical form, making it difficult to see the common structure behind them. Many theorems appear different because their contractions use rational expressions, multivalued operators, partial orders or complex-valued distances. However, most of them depend on three shared components: a complete generalized distance space, a contraction condition strong enough to control the Picard sequence, and a uniqueness condition preventing multiple limit points. The study therefore asks how these components work together and what conceptual role is played by the b-metric coefficient and complex-valued ordering.

1.2 Objectives of the Study

- To explain the mathematical meaning of b-metric and complex-valued b-metric spaces.
- To compare ordinary metric, b-metric, complex-valued metric and complex-valued b-metric structures.
- To analyse how generalized contractions produce existence and uniqueness of fixed points.
- To interpret the role of the b-metric coefficient s in convergence estimates.
- To show how rational and weak contraction forms extend classical contraction arguments.

2. Conceptual Background

2.1 Fixed Point Theory and Generalized Distances

Fixed point theory begins with the idea that repeated application of a suitable mapping may converge to a stable point. If x_0 is an initial point and $x_{n+1} = Tx_n$, the sequence $\{x_n\}$ is called a Picard sequence. In a complete setting, if the distances between successive iterates become sufficiently small and the sequence is Cauchy, the limit point x^* may satisfy $Tx^* = x^*$. In classical metric spaces this mechanism is relatively direct, but in b-metric spaces the triangle inequality contains the coefficient s. This coefficient appears repeatedly in the proof and therefore affects the contraction bounds required for convergence.

The move from metric spaces to b-metric spaces reflects a broader trend in nonlinear analysis: rather than forcing a problem into a rigid metric framework, researchers modify the distance structure so that it better fits the problem. Boriceanu (2009) used vector-valued b-metrics to accommodate systems in which components have different distance behaviour. Aydi et al. (2012) demonstrated that set-valued quasi-contractions can be studied within b-metric settings. These studies illustrate the usefulness of generalized distance structures for extending fixed point arguments.

2.2 B-Metric Spaces

Let X be a non-empty set and $s \geq 1$. A function $d: X \times X \to [0, \infty)$ is called a b-metric if, for all x, y, z in X: (i) $d(x,y) = 0$ if and only if $x = y$; (ii) $d(x,y) = d(y,x)$; and (iii) $d(x,z) \leq s[d(x,y) + d(y,z)]$. The pair (X,d) is then called a b-metric space. When $s = 1$, the definition reduces to the ordinary metric case. When $s > 1$, the space allows a controlled relaxation of the triangle inequality.

A typical example is $d(x,y) = |x-y|^p$ on R for $p > 1$. This function does not generally satisfy the ordinary triangle inequality, but it can satisfy a b-metric inequality with an appropriate coefficient. Such examples show why b-metric spaces are useful: they include distance-like functions that are natural in analysis but not metrics in the classical sense. Berinde and Păcurar (2022) emphasize that the b-metric literature has developed rapidly and that precise referencing and careful formulation are necessary because many results depend delicately on the coefficient and completeness assumptions.

2.3 Complex-Valued B-Metric Spaces

Complex-valued metric spaces were introduced to allow distances taking values in the complex plane under a partial order. A common order on complex numbers is defined by $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Azam et al. (2011) used this framework to prove fixed point results involving rational inequalities. A complex-valued b-metric space further modifies the triangle inequality by adding the coefficient $s \geq 1$. Thus, the distance is not only complex-valued but also b-metric in structure.

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In a complex-valued b-metric space, the expression d(x,y) belongs to C, but its comparison is interpreted through the order of real and imaginary components. Rao et al. (2013) developed a common fixed point theorem in this setting, and later studies by Mukheimer (2014), Singh et al. (2015), Bharadwaj et al. (2016) and Saadi and Hamaizia (2023) extended the theory using rational, common and multivalued contraction conditions. The complex-valued setting is analytically useful because it can represent two-component distance information, though it also requires careful handling of order, convergence and modulus.

Table 1. Comparison of metric-type structures used in the paper

Structure	Distance range	Triangle condition	Analytical significance
Metric space	d: X x X -> [0, infinity)	d(x,z) <= d(x,y)+d(y,z)	Classical complete metric fixed point theory
B-metric space	d: X x X -> [0, infinity)	d(x,z) <= s[d(x,y)+d(y,z)], s >= 1	Relaxes triangle inequality and changes contraction bounds
Complex-valued metric space	d: X x X -> C	d(x,z) <= d(x,y)+d(y,z) under complex order	Allows ordered complex distance and rational contractions
Complex-valued b-metric space	d: X x X -> C	d(x,z) <= s[d(x,y)+d(y,z)] under complex order	Combines complex-valued distance with b-metric coefficient

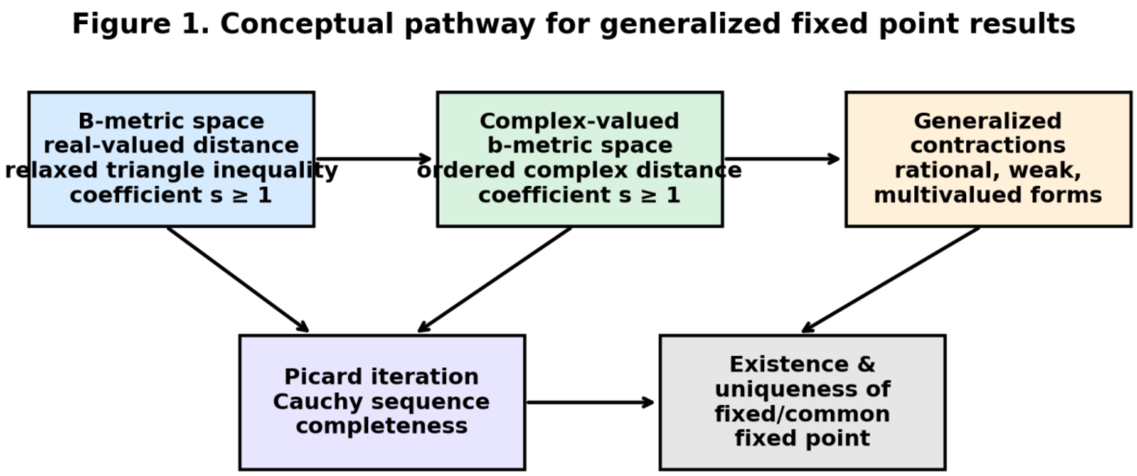

Figure 1. Conceptual pathway for generalized fixed point results.

3. Review of Literature

The literature on generalized fixed point theorems in b-metric and complex-valued b-metric spaces may be grouped into four broad streams: foundational development of b-metric theory, generalized contractions in b-metric spaces, complex-valued metric and b-metric extensions, and recent multivalued or extended versions. Each stream contributes to the way modern fixed point theorems are formulated.

Boriceanu (2009) made an important contribution by studying vector-valued b-metrics and showing how fixed point methods can be adapted to settings where distance is not simply scalar. This approach is important because many systems of equations naturally have componentwise behaviour. Aydi et al. (2012) then developed a fixed point theorem for set-valued quasi-contractions in b-metric spaces. Their work is significant because multivalued mappings are common in optimization, differential inclusions and dynamic processes. In related work, Aghajani et al. (2014) examined generalized weak contractive mappings in partially ordered complete b-metric spaces, thereby connecting order structures with b-metric fixed point theory.

Another important branch concerns complex-valued metric spaces. Azam et al. (2011) introduced complex-valued metric spaces and established common fixed point results under rational contractive inequalities. Rouzkard and Imdad (2012) extended this direction using rational expressions in complex-valued metric spaces. Nashine et al. (2014) continued the rational contraction approach, while Sitthikul and Saejung (2012) weakened some assumptions through point-dependent contractive conditions. These works created the conceptual base for complex-valued b-metric spaces, where the b-metric coefficient adds an additional layer of generality.

Rao et al. (2013) introduced complex-valued b-metric spaces and proved a common fixed point theorem in that setting. Mukheimer (2014) then established common fixed point theorems for mappings in complex-valued b-metric spaces, and Singh et al. (2015) proved common fixed point theorems for pairs of mappings satisfying rational contractions. Bharadwaj et al. (2016) further organized complex-valued b-metric fixed point theorems and presented useful definitions, examples and corollaries. More recently, Saadi and Hamaizia (2023) developed multivalued common fixed point results in complex b-metric spaces, demonstrating that the field continues to move toward broader and more flexible operator frameworks.

Recent work has also extended b-metric theory beyond the constant-coefficient framework. Singh and Ghosh (2025) studied extended b-metric spaces using rational-type contractions and provided common fixed point results for more than one mapping. Such research indicates that the future of generalized fixed point theory will likely involve hybrid structures combining b-metric coefficients, control functions, partial orders, graph structures, complex values and multivalued operators. The central challenge is to retain enough structure to guarantee convergence while allowing sufficient generality to model nonlinear problems.

Table 2. Thematic synthesis of selected literature from 2009 to 2025

Author(s)	Main focus	Contribution to this study
Boriceanu (2009)	Vector-valued b-metrics	Extended fixed point theory to componentwise distance frameworks.
Aydi et al. (2012)	Set-valued quasi-contractions	Generalized multivalued fixed point results in b-metric spaces.
Azam et al. (2011)	Complex-valued metric spaces	Introduced complex-valued metric setting for common fixed points.
Rao et al. (2013)	Complex-valued b-metric spaces	Combined complex-valued distance with b-metric relaxation.
Mukheimer (2014)	Common fixed points	Proved common fixed point results in complex-valued b-metric spaces.
Saadi & Hamaizia (2023)	Multivalued fixed points	Extended complex b-metric results to multivalued mappings.
Singh & Ghosh (2025)	Extended b-metrics	Developed rational-type common fixed point theorems in extended b-metric spaces.

4. Research Methodology

4.1 Research Design

This study uses a descriptive, analytical and doctrinal-mathematical research design. It is descriptive because it explains the concepts of b-metric spaces, complex-valued b-metric spaces and generalized contractions. It is analytical because it compares the assumptions and conclusions of selected fixed point theorems. It is doctrinal-mathematical because it examines definitions, theorem structures and proof logic in previously published mathematical literature.

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4.2 Sources and Selection Criteria

The paper relies on secondary scholarly sources published between 2009 and 2025. Only author-based academic references are used in order to avoid institutional, departmental or encyclopedic sources. The selected works were chosen because they represent major conceptual steps in b-metric and complex-valued b-metric fixed point theory: vector-valued b-metrics, set-valued contractions, complex-valued metric spaces, complex-valued b-metric spaces, rational contractions, multivalued common fixed points and extended b-metric spaces.

4.3 Analytical Procedure

The analysis follows four steps. First, the basic definitions of metric-type spaces are compared. Second, the contraction conditions are classified according to their structure: simple, rational, weak, multivalued or common fixed point form. Third, the role of the b-metric coefficient is interpreted through convergence estimates and illustrative graphs. Fourth, a conceptual theorem schema is formulated to show the shared pattern across generalized fixed point results. This procedure helps convert technical literature into an integrated research framework.

Table 3. Methodological framework of the study

Element	Description	Purpose
Research type	Descriptive, analytical and doctrinal-mathematical	To explain and compare fixed point structures
Data sources	Author-based scholarly articles from 2009-2025	To comply with reference requirements and ensure academic basis
Unit of analysis	Theorem structure and contraction condition	To identify common proof logic
Visual method	Tables, conceptual diagram and convergence graphs	To clarify abstract mathematical relations
Limitation	No new theorem proof claimed	The study is a conceptual synthesis, not an original technical proof paper

5. Theoretical Analysis

5.1 Role of the B-Metric Coefficient

The coefficient s is the defining feature of a b-metric space. It measures how much the triangle inequality is allowed to expand. In the metric case, the distance from x to z is bounded directly by the sum of the distances through y. In a b-metric space, the sum is multiplied by s. This relaxation increases flexibility but also weakens the estimates used in fixed point proofs. As a result, the admissible contraction constant is often smaller than in the classical metric case. In many b-metric results, one encounters conditions such as $q < 1/s$, $q < 1/s^2$, or related restrictions depending on the theorem structure.

This restriction is not merely technical. If the contraction constant is too large compared with the coefficient, the Picard sequence may not be sufficiently controlled. The proof usually requires bounding $d(x_n, x_m)$ by a series whose terms decrease geometrically. The b-metric coefficient appears when applying the relaxed triangle inequality repeatedly. Thus, convergence depends on the balance between the contraction strength q and the coefficient s. This explains why generalized b-metric theorems must be formulated with attention to the coefficient.

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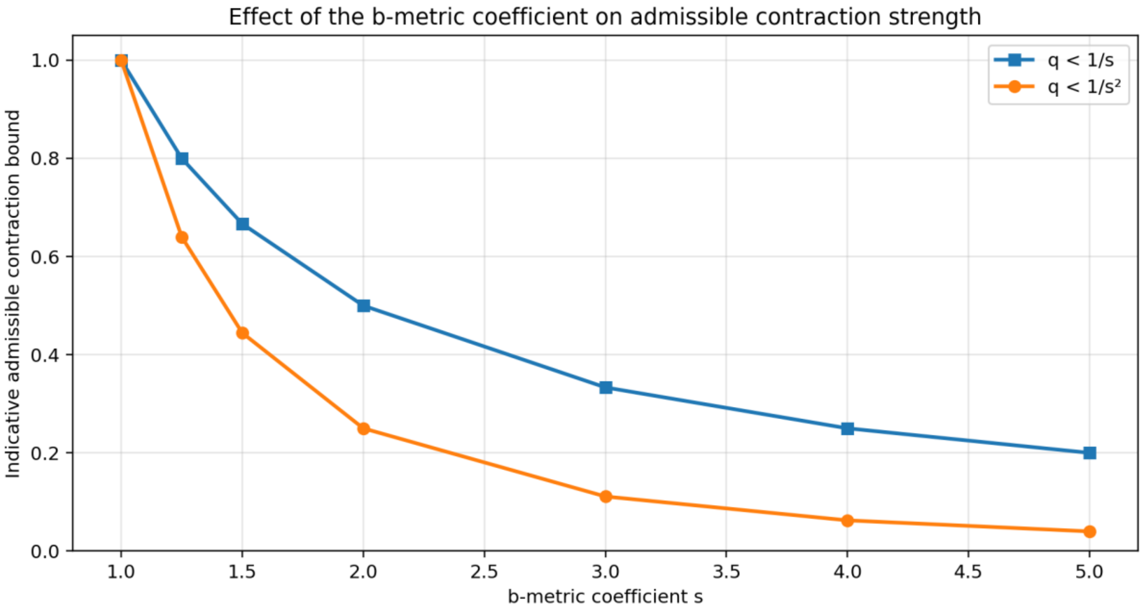

Figure 2. Effect of the b-metric coefficient on admissible contraction strength.

5.2 Generalized Contractions

A contraction is generalized when it modifies the simple inequality d(Tx,Ty) <= q d(x,y). In b-metric and complex-valued b-metric spaces, generalized contractions may include additional terms such as d(x,Tx), d(y,Ty), d(x,Ty), rational expressions, control functions, weak comparison functions, or multivalued Hausdorff-type distances. The aim is to capture nonlinear relationships that cannot be represented by a single constant q.

Rational contractions are particularly common in complex-valued fixed point theory. They include expressions where products of distances are divided by terms such as 1 + d(x,y). These forms help control mappings that are not strict Banach contractions but still generate convergent iterates. Mukheimer (2014) and Singh et al. (2015) use rational-type conditions in complex-valued b-metric spaces, while Singh and Ghosh (2025) use rational-type contractions in extended b-metric spaces. The shared logic is that the contractive expression must still force the iterative distance to decrease in a controlled manner.

Table 4. Common generalized contraction patterns

Type	Typical form	Interpretation
Banach-type	d(Tx,Ty) <= q d(x,y)	q must be sufficiently small; in b-metric results the coefficient s affects q
Weak contraction	d(Tx,Ty) <= d(x,y) - phi(d(x,y))	A positive altering function phi creates strict decrease
Rational contraction	Inequality involving products and quotients of distances	Useful for nonlinear expressions not covered by simple contraction
Common fixed point form	Conditions on Sx and Ty or pairs of mappings	Used for two or more mappings sharing one fixed point
Multivalued contraction	Distance between Tx and Ty uses set-valued metric relation	Models operators with multiple possible images

5.3 Complex Ordering and Convergence

In complex-valued b-metric spaces, the distance d(x,y) is a complex number. To compare distances, a partial order is imposed on complex numbers. Usually z1 <= z2 means that both the real part and the imaginary part of z1 are less than or equal to the corresponding parts of z2. This order allows inequalities such as d(Sx,Ty) <= k d(x,y) to be meaningful. However, because the complex plane is not totally ordered, not every pair of complex values is comparable. This creates a limitation: proofs must ensure that the relevant distances are comparable under the selected order.

Convergence in complex-valued b-metric spaces is commonly handled by connecting the complex-valued distance to its modulus. If |d(x_n,x)| tends to zero, the sequence is interpreted as convergent. This reduces some difficulties of complex ordering, but the proof still depends on complex-valued inequalities. Bharadwaj et al. (2016) explicitly discuss definitions of convergence, Cauchy sequences and completeness in complex-valued b-metric spaces. The central point is that completeness remains essential: without it, a Cauchy Picard sequence may fail to converge within the space.

5.4 Common Fixed Point Theorem Schema

Across many fixed point results in the literature, the theorem structure can be expressed as a common schema. Let (X,d) be a complete b-metric or complex-valued b-metric space, and let S,T: X -> X be self-mappings. Suppose the pair (S,T) satisfies a contractive condition strong enough to imply that the generated iterative sequence is Cauchy. Suppose also that the space is complete and the mappings satisfy any required compatibility, continuity, weak compatibility or admissibility conditions. Then S and T have a common fixed point. If an additional uniqueness condition holds, that fixed point is unique.

This schema does not replace individual theorems. Rather, it identifies their shared architecture. The differences between theorems lie in the exact contraction expression, the mapping type and the auxiliary assumptions. For example, Aydi et al. (2012) focus on set-valued quasi-contractions, Aghajani et al. (2014) use generalized weak contractions in an ordered setting, and Saadi and Hamaizia (2023) use multivalued common fixed point conditions in complex b-metric spaces. Nevertheless, each result still follows the same broad logic: construct an iteration, show it is Cauchy, use completeness to obtain a limit, and verify that the limit is a fixed point.

6. Illustrative Example and Graphical Interpretation

6.1 Example of a B-Metric Space

Consider X = [0,1] and define d(x,y) = |x-y|^2. This distance is not generally a classical metric because it may fail the ordinary triangle inequality. However, it is a b-metric with coefficient s = 2. The example demonstrates the core value of b-metric theory: it permits distance functions that are analytically meaningful even when the strict metric triangle inequality fails. If a mapping T on X satisfies an appropriate contraction condition relative to this b-metric, a fixed point theorem can still be applied.

For a simple illustrative mapping, let T(x) = x/3 on X. The fixed point is 0 because T(0) = 0. Starting with x0 = 1, the Picard sequence is x_n = (1/3)^n. Under d(x,y) = |x-y|^2, the distance to the fixed point becomes d(x_n,0) = (1/9)^n. This shows rapid convergence. In more complicated theorems, the sequence may be controlled through rational or weak contraction terms, but the underlying idea remains the same.

Table 5. Illustrative Picard iteration for T(x)=x/3 under d(x,y)=|x-y|²

n	x_n	d(x_n, 0)
0	1.000000	1.00000000
1	0.333333	0.11111111
2	0.111111	0.01234568
3	0.037037	0.00137174
4	0.012346	0.00015242
5	0.004115	0.00001694
6	0.001372	0.00000188
7	0.000457	0.00000021

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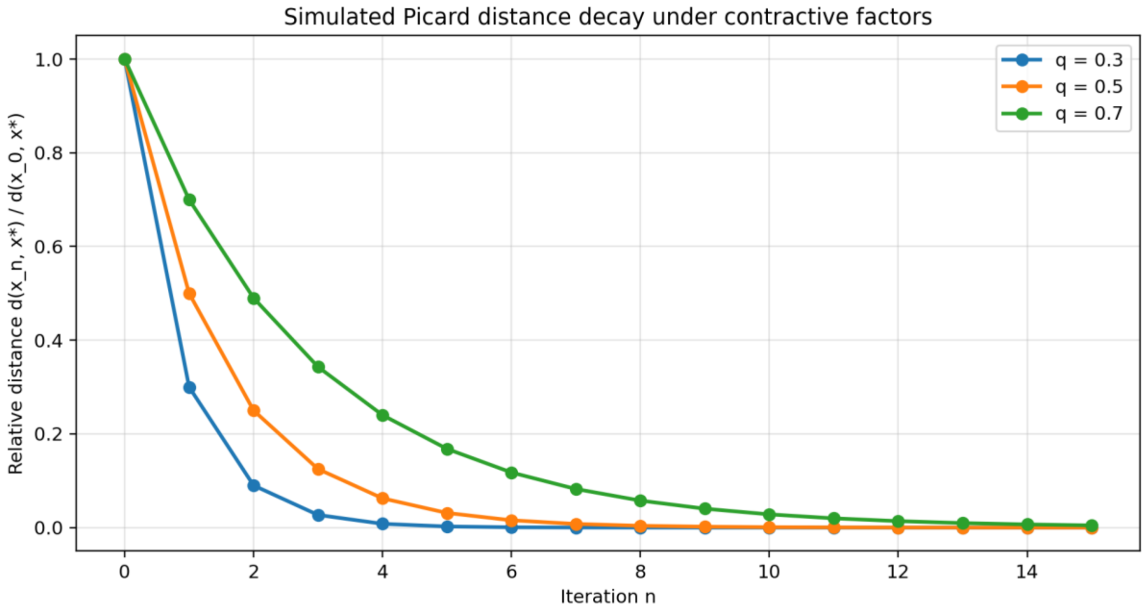

Figure 3. Simulated distance decay for Picard iteration under different contractive factors.

6.2 Interpretation of the Graphs

Figure 2 shows that as the b-metric coefficient s increases, the admissible contraction strength must usually become stricter. Figure 3 illustrates the familiar idea that smaller contractive factors produce faster convergence. These graphs are not intended to prove a theorem; they are explanatory visualizations. They help connect abstract proof conditions with intuitive behaviour of iterations. In fixed point theory, convergence speed is not the only concern, because existence and uniqueness are logically prior. However, the graphs show why strong contraction conditions are important in generalized spaces.

7. Applications of Generalized Fixed Point Theorems

Generalized fixed point theorems in b-metric and complex-valued b-metric spaces are applicable in areas where ordinary metric assumptions are too restrictive. One major area is integral equations. A nonlinear integral equation can often be transformed into an operator equation Tx = x. If the operator satisfies a generalized contraction in a suitable complete b-metric space, a fixed point theorem provides existence and sometimes uniqueness of the solution. This is one reason fixed point theory remains important in nonlinear functional analysis.

A second application area is systems of equations. Vector-valued b-metric spaces and complex-valued distances are useful when a problem has more than one component of error. For example, an iterative numerical method may have two types of residuals or two coupled variables. A complex-valued distance can encode two components through real and imaginary parts, while a vector-valued b-metric can treat components separately. Boriceanu (2009) is relevant here because vector-valued b-metrics make it possible to study systems where each component has its own contractive behaviour.

A third application area is dynamic programming and optimization. Many dynamic processes can be formulated as fixed point problems involving operators on function spaces. When the operator is not a simple contraction in the ordinary metric, generalized conditions may still guarantee a fixed point. Rational contractions and weak contractions are useful because they allow nonlinear dependence on both the current distance and the residual distances d(x,Tx) and d(y,Ty). This flexibility is essential in applications where the operator is contractive only after incorporating additional information.

Multivalued fixed point results are also important. In optimization and differential inclusions, a point may be mapped not to a single value but to a set of possible values. Aydi et al. (2012) and Saadi and Hamaizia (2023) are significant because they extend fixed point reasoning to set-valued and multivalued settings. These theorems can support the existence of solutions where deterministic single-valued maps are insufficient.

8. Discussion

8.1 Strengths of B-Metric Generalization

The main strength of b-metric generalization is flexibility. It allows researchers to work with distance functions that arise naturally but do not satisfy the ordinary triangle inequality. This is useful in functional analysis, approximation theory, nonlinear equations and applied mathematical modelling. The coefficient s provides a controlled relaxation rather than an unrestricted weakening of distance. Therefore, b-metric spaces retain enough structure to support convergence while enlarging the class of admissible spaces.

Another strength is compatibility with further extensions. B-metric spaces can be combined with partial orders, graphs, cone structures, vector values, multivalued mappings and complex-valued distances. This makes the theory modular. A researcher can adapt the distance structure to the problem and then formulate contractive assumptions accordingly. The literature from 2009 to 2025 shows a steady movement from basic b-metric contractions toward more complex hybrid frameworks.

8.2 Challenges and Limitations

The first limitation is that not every formal generalization produces a genuinely new mathematical result. Some results may be reducible to known metric theorems after transforming the b-metric into an equivalent metric or using the modulus of a complex-valued distance. This has led to debates in the literature about whether certain generalized theorems provide real novelty. Berinde and Păcurar (2022) warn that careful attention to historical development and exact assumptions is necessary in b-metric research.

The second limitation concerns proof validity. In b-metric spaces, repeated use of the relaxed triangle inequality can multiply constants in ways that are easy to underestimate. A proof that works in a metric space may fail in a b-metric space if the coefficient s is not correctly handled. Similarly, in complex-valued b-metric spaces, order comparability and convergence definitions must be treated carefully. A complex-valued inequality is meaningful only under the chosen partial order, and the proof must ensure that relevant terms are comparable.

The third limitation concerns applications. Many papers include abstract examples, but fewer develop substantial applications to concrete operator equations, numerical schemes or boundary value problems. Future research should strengthen the link between theorem development and applied problem solving. The most valuable generalized fixed point results will be those that solve problems not easily handled by classical metric results.

8.3 Synthesis of Key Findings

Table 6. Key findings of the study

Finding	Explanation
Completeness remains essential	A Cauchy Picard sequence must converge inside the space.
The coefficient s controls proof strength	Larger s usually requires stricter contraction conditions.
Complex-valued distances increase expressiveness	Real and imaginary parts can represent two-component distance information.
Rational contractions expand applicability	They handle nonlinear dependence beyond simple Banach contraction.
Multivalued results widen application scope	They are useful for inclusions, optimization and nondeterministic operators.
Novelty requires careful verification	Some generalized results may reduce to older metric results if assumptions are not examined.

9. Conclusion

This paper has examined generalized fixed point theorems in b-metric and complex-valued b-metric spaces as a conceptual and analytical framework. The study shows that b-metric spaces generalize metric spaces by relaxing the triangle inequality through a coefficient $s >= 1$. This coefficient is not a minor technical detail; it directly influences convergence estimates and admissible contraction bounds. Complex-valued b-metric spaces add another layer of generality by allowing distances to take complex values under a partial order. Together, these structures enable fixed point theorems to address a wider class of nonlinear and multicomponent problems.

The literature reviewed in this paper shows a clear progression from vector-valued and set-valued b-metric results to complex-valued, rational, common and multivalued fixed point theorems. The common proof architecture remains stable: define an iterative sequence, prove that the sequence is Cauchy under a suitable contraction, use completeness to obtain a limit, and verify that the limit is a fixed point. However, each generalization introduces additional technical demands. In b-metric spaces, the coefficient must be carefully controlled. In complex-valued b-metric spaces, the order relation and modulus-based convergence must be handled precisely.

The paper concludes that generalized fixed point theory is valuable when it increases the ability to model and solve nonlinear problems that classical metrics cannot handle naturally. At the same time, researchers should avoid unnecessary generalization and should demonstrate how new results improve, extend or apply beyond known theorems. The strongest future work will combine rigorous proof, clear examples and meaningful applications to equations, inclusions, optimization and dynamic systems.

10. Recommendations for Future Research

- Future studies should state the role of the b-metric coefficient explicitly in every contraction condition.
- Research should include examples that cannot be reduced easily to ordinary metric fixed point theorems.
- Complex-valued b-metric results should clearly justify the order relation and convergence definition used.
- More applications should be developed for nonlinear integral equations, differential inclusions and dynamic programming problems.
- Comparative studies should evaluate whether generalized theorems offer sharper or broader conclusions than existing results.
- Researchers should develop computational illustrations showing how Picard iteration behaves under different coefficients and contractions.

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