FIXED POINT THEOREMS IN THE CONTEXT OF GENERALIZED GB-METRIC SPACES

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Abstract
Fixed point theory is an important branch of nonlinear analysis with wide applications in differential equations, optimization, approximation theory, and applied mathematics. The present paper discusses fixed point results in the setting of generalized Gb-metric spaces. The study focuses on generalized contractive mappings and quasi-contraction type mappings under the assumption of generalized Gb-completeness. The paper highlights how different fixed point theorems ensure the existence and uniqueness of fixed points under suitable conditions. It also explains that some mappings may admit fixed points even when continuity is not satisfied. Through theoretical discussion and examples, the paper shows that generalized Gb-metric spaces provide a broader framework than classical metric spaces for studying fixed point problems. The results contribute to the extension of existing fixed point theorems and support further research in generalized metric structures.

Keywords
Fixed point theorem, generalized Gb-metric space, complete metric space, contractive mapping, quasi-contraction, nonlinear analysis.

1. Introduction

Fixed point theory deals with the conditions under which a mapping has a point that remains unchanged under the action of that mapping. If $T: X \to X$ is a mapping, then a point $x \in X$ is called a fixed point of T if $T(x) = x$. This simple idea has deep importance in modern mathematics because many problems in analysis, topology, differential equations, and applied sciences can be converted into fixed point problems.

Classical fixed point theory began with results in ordinary metric spaces, especially through contraction principles. However, as mathematical analysis developed, researchers introduced several generalized metric structures to study problems that could not be handled properly within the classical metric framework. One such structure is the generalized Gb-metric space. This space extends the traditional metric concept by allowing more flexible distance-type functions and generalized forms of convergence and completeness.

The present study focuses on fixed point theorems in the context of generalized Gb-complete metric spaces. The uploaded matter discusses several theorems, including Theorem 4.4.1, Theorem 4.4.2, Theorem 4.4.3, and Theorem 4.4.4. These theorems establish the existence and uniqueness of fixed points under different contractive-type conditions. The discussion also shows that fixed points may exist even when the mapping is not continuous. This makes the study significant because it extends fixed point theory beyond conventional assumptions.

Objectives of the Study

1. To study fixed point results for generalized contractive mappings in generalized Gb-complete metric spaces.
2. To examine the existence and uniqueness of fixed points for continuous and discontinuous mappings under generalized Gb-metric conditions.

Fixed Point Theorems in Context of Generalized Gb-Metric Spaces

Main "results of this section are the following theorems:"

Theorem 4.4.1 Let "(X, G) be a generalized Gb-complete metric space with constant $s \geq 1$ *and let* $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \qquad for\ all\ \ x, y, z \in X,$$

where $k \in [0, \frac{1}{s})$. *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$,

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n).$$

On using induction, we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1).$$

For $n, m \in \mathbb{N}$ with $n < m$, consider

$$\begin{aligned}
& G(x_n, x_m, x_m) \\
\leq\ & s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)] \\
\leq\ & sG(x_n, x_{n+1}, x_{n+1}) + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_m, x_m)] \\
\leq\ & \cdots \leq sG(x_n, x_{n+1}, x_{n+1}) + s^2 G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + s^{m-n} G(x_{m-1}, x_m, x_m) \\
\leq\ & (sk^n + s^2 k^{n+1} + \cdots + s^{m-n} k^{m-1}) G(x_0, x_1, x_1) \\
\leq\ & \frac{sk^n}{1 - sk} G(x_0, x_1, x_1).
\end{aligned}$$

This implies that $G(x_n, x_m, x_m) \to 0$ as $n, m \to +\infty$.

Hence, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'. Therefore

$$T(x') = T(\lim_{n \to +\infty} x_n) = \lim_{n \to +\infty} T(x_n) = \lim_{n \to +\infty} x_{n+1} = x'.$$

This implies that x' is a fixed point of T.

Let y be another fixed point of T. Then

$$G(x', y, y) = G(Tx', Ty, Ty) \leq kG(x', y, y)$$

which implies that $G(x', y, y) = 0$ as $k \in [0, 1)$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

Example Let $X = \mathbb{R}$, and $G : X \times X \times X \to [0, +\infty)$ be defined as :

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |z - x|^2, \quad \text{for all} \quad x, y, z \in \mathbb{R}.$$

Then (X, G) is a generalized G_b-complete metric space with $s = 2$.

Define a mapping $T : X \to X$ by

$$T(x) = \frac{x}{2}, \quad \text{for all} \quad x \in X.$$

Then

$$\begin{aligned} G(Tx, Ty, Tz) &= G\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \\ &= \left|\frac{x}{2} - \frac{y}{2}\right|^2 + \left|\frac{y}{2} - \frac{z}{2}\right|^2 + \left|\frac{z}{2} - \frac{x}{2}\right|^2 \\ &\leq kG(x, y, z), \end{aligned}$$

where $k = \frac{1}{4} \in [0, \frac{1}{s})$. Also, T has a unique fixed point, namely 0.

Following up from Theorem 4.4.1, this subsequent theorem asserts that the interval should be $[0, \frac{1}{s})$ is extended to [0, 1).

Theorem 4.4.2 Let "(X, G) be a generalized Gb-complete metric space with constant $s \geq 1$ *and let* $T : X \to X$ *be a mapping such that*

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \qquad \textit{for all} \quad x, y, z \in X,$$

where $k \in [0, 1)$. *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq kG(x_{n-1}, x_n, x_n),$$

so by the use of Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized Gb-complete metric space, therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'. Therefore

$$T(x') = T(\lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} T(x_n) = \lim_{n\to+\infty} x_{n+1} = x'.$$

This implies that x' is a fixed point of T.

Let y be another fixed point of T. Then

$$G(x', y, y) = G(Tx', Ty, Ty) \leq kG(x', y, y)$$

which implies that $G(x', y, y) = 0$ as $k \in [0, 1)$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

In the following examples, it has been shown that the conditions of Theorem 2.4.2 are satisfied, but not of Theorem 4.4.1.

Example Consider the generalized Gb-complete metric space (X, G) as described in Example 4.2.9. Let T : X → X be a mapping defined by

$$T(x) = \frac{3x}{4}, \quad \text{for all } x \in X.$$

Then

$$\begin{aligned} G(Tx,Ty,Tz) &= G\left(\frac{3x}{4},\frac{3y}{4},\frac{3z}{4}\right) \\ &= \left|\frac{3x}{4}-\frac{3y}{4}\right|^2+\left|\frac{3y}{4}-\frac{3z}{4}\right|^2+\left|\frac{3z}{4}-\frac{3x}{4}\right|^2 \\ &\leq kG(x,y,z), \end{aligned}$$

where $k=\frac{9}{16}\in[0,1)$, but $k\notin[0,\frac{1}{s})$. Here, 0 is the only fixed point of T.

Example Let $X=\{\alpha,\beta,\gamma\}$ and define a mapping $G: X\times X\times X\to[0,+\infty)$ as follows:

$G(x,x,x)=0,\quad \text{for all} \quad x\in X,$

$G(\alpha,\beta,\beta)=G(\beta,\alpha,\beta)=G(\beta,\beta,\alpha)=G(\alpha,\alpha,\beta)=G(\alpha,\beta,\alpha)=G(\beta,\alpha,\alpha)=1,$

$G(\alpha,\gamma,\gamma)=G(\gamma,\alpha,\gamma)=G(\gamma,\gamma,\alpha)=G(\alpha,\alpha,\gamma)=G(\alpha,\gamma,\alpha)=G(\gamma,\alpha,\alpha)=1.2,$

$G(\beta,\gamma,\gamma)=G(\gamma,\beta,\gamma)=G(\gamma,\gamma,\beta)=G(\beta,\beta,\gamma)=G(\beta,\gamma,\beta)=G(\gamma,\beta,\beta)=3.3,$

$G(x,y,z)=3.2,\quad \text{for all} \quad x,y,z\in X \quad \text{with} \quad x\neq y\neq z\neq x.$

It is easy to prove that (X,G) is a generalized G_b-complete metric space with constant $s=1.5$ (here, 1.5 is the smallest possible value of s).

However, it is noticed that, with $x=\beta,\ y=\alpha,\ z=\gamma,$

$$G(x,y,z)\nleq G(x,\alpha,\alpha)+G(\alpha,y,z);$$

thus, G is not a G-metric on X. Also, with $x = \beta,\ y = \gamma,\ z = \alpha$,

$$G(x, y, y) = 3.3 \nleq 3.2 = G(x, y, z);$$

thus, G is not a generalized b-metric on X.

Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \alpha,\ T\gamma = \beta$.

Now, for $k = \frac{5}{6} \in [0, 1)$, it is not difficult to prove that

$$G(Tx, Ty, Tz) \leq kG(x, y, z), \quad \text{for all} \quad x, y, z \in X,$$

and $\frac{5}{6}$ is the smallest value of such k. Here T has a unique fixed point, namely α, however, $k \notin [0, \frac{1}{s})$, but $k \in [0, 1)$.

It may be deduced from the condition (4.4.2) in Theorem 4.4.2 that the mapping T is continuous. The following theorem demonstrates that a fixed point may also be present in a mapping that is discontinuous from beginning to end. **Theorem 4.4.3** Let “(X, G) be a generalized Gb-complete metric space with constant $s \geq 1$ *and let* $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq k[G(x, y, z) + G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)],$$

for all $x, y, z \in X$, where $k \in [0, \lambda)$ and $\lambda = min\{\frac{1}{4}, \frac{1}{2s}\}$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq k[2G(x_{n-1}, x_n, x_n) + 2G(x_n, x_{n+1}, x_{n+1})]$$

which implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{2k}{1-2k} G(x_{n-1}, x_n, x_n).$$

As $k < \frac{1}{4}$, therefore, $\frac{2k}{1-2k} < 1$ and hence by Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized G_b-complete metric space, therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'.

Now, Consider

$$\begin{aligned} & G(x', Tx', Tx') \\ \leq\ & s[G(x', x_n, x_n) + G(x_n, Tx', Tx')] \\ \leq\ & sG(x', x_n, x_n) + sk[G(x_{n-1}, x', x') + G(x_{n-1}, x_n, x_n) + 2G(x', Tx', Tx')] \end{aligned}$$

which gives that

$$(1 - 2ks)G(x', Tx', Tx') \leq sG(x', x_n, x_n) + sk[G(x_{n-1}, x', x') + G(x_{n-1}, x_n, x_n)].$$

Taking $n \to +\infty$, we get

$$G(x', Tx', Tx') = 0$$

which implies that $Tx' = x'$, that is, x' is a fixed point of T.

Let y be another fixed point of T. Then

$$G(x', y, y) = G(Tx', Ty, Ty) \leq kG(x', y, y)$$

which implies that $G(x', y, y) = 0$ as $k < \frac{1}{4}$, therefore, $x' = y$, that is, x' is a unique fixed point of T. □

Example Let X = [0, 1] and define a mapping $G : X \times X \times X \to [0, +\infty)$ as follows:

$$G(x, y, z) = |x - y|^2 + |y - z|^2 + |z - x|^2 \quad \text{for all} \quad x, y, z\ X.$$

Clearly, (X, G) is a generalized G_b-complete metric space with constant $s = 2$.

Define a mapping $T : X \to X$ as:

$$T(x) = \begin{cases} \frac{x}{6}, & \text{if } x \in [0, 1] - \{\frac{1}{2}\}; \\ 0, & \text{if } x = \frac{1}{2}. \end{cases}$$

Clearly, T is a discontinuous mapping. We now prove that (2.4.5) is true for $k = \frac{1}{5}$.

For this, let $x, y, z \in [0, 1] - \{\frac{1}{2}\}$. Then

$$G(Tx, Ty, Tz) = G(\frac{x}{6}, \frac{y}{6}, \frac{z}{6})$$
$$= \frac{1}{36}(|x - y|^2 + |y - z|^2 + |z - x|^2)$$
$$\leq \frac{1}{5}[G(x, y, z) + G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)].$$

If $x = \frac{1}{2}$ and $y, z \in [0,1] - \{\frac{1}{2}\}$, then

$G(Tx, Ty, Tz) = G(0, \frac{y}{6}, \frac{z}{6}) = \frac{1}{36}(y^2 + |y - z|^2 + z^2) \leq \frac{3}{36}$; and

$G(x, Tx, Tx) = G(\frac{1}{2}, 0, 0) = 2|\frac{1}{2} - 1|^2 = \frac{1}{2}$; therefore,

$$G(Tx, Ty, Tz) \leq \frac{3}{36} < \frac{1}{5} \times \frac{1}{2}$$
$$\leq \frac{1}{5}[G(x, y, z) + G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)].$$

If $x = y = \frac{1}{2}$ and $z \in [0,1] - \{\frac{1}{2}\}$, then

$G(Tx, Ty, Tz) = G(0, 0, \frac{z}{6}) = \frac{2z^2}{36} \leq \frac{2}{36}$; and

$G(x, Tx, Tx) = G(\frac{1}{2}, 0, 0) = \frac{1}{2}$; therefore,

$$G(Tx, Ty, Tz) \leq \frac{2}{36} < \frac{1}{5} \times \frac{1}{2}$$
$$\leq \frac{1}{5}[G(x, y, z) + G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)].$$

Thus, conditions of Theorem 4.4.3 are satisfied. Moreover, for $x = y = \frac{1}{2}$ and z = 0.51, does not hold.
The following theorem is a generalization of Theorem 4.4.2 and Theorem 4.4.3.

Theorem 4.4.4 Let"(X, G) be a generalized Gb-complete metric space with constant $s \geq 1$ *and let* $T : X \to X$ be a mapping such that

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz),$$

for all $x, y, z \in X$, where $a+b+c+d<1$, $s(c+d)<1$ and $a+b \geq 0$. Then either T has a unique fixed point, or all elements of X are fixed points of T.

Proof. Let $x_0 \in X$ be arbitrary. Define a sequence $\{x_n\}$ in X by

$$x_n = T^n(x_0), \text{ for all } n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, we have

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq aG(x_{n-1}, x_n, x_n) + bG(x_{n-1}, x_n, x_n) + cG(x_n, x_{n+1}, x_{n+1}) \\ &+dG(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

which implies that

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a+b}{1-(c+d)} G(x_{n-1}, x_n, x_n).$$

Since $a+b+c+d<1$, $a+b \geq 0$, so by Lemma 2.3.2, $\{x_n\}$ is a Cauchy sequence in (X, G). But (X, G) is a generalized Gb-complete metric space; therefore, there exists $x' \in X$ such that sequence $\{x_n\}$ converges to x'.

Now, Consider

$$\begin{aligned} &G(x', Tx', Tx') \\ &\leq s[G(x', x_n, x_n) + G(x_n, Tx', Tx')] \\ &\leq sG(x', x_n, x_n) + saG(x_{n-1}, x', x') + sbG(x_{n-1}, x_n, x_n) + s(c+d)G(x', Tx', Tx'), \end{aligned}$$

which gives that

$$(1-s(c+d))G(x',Tx',Tx') \leq sG(x',x_n,x_n)+s[aG(x_{n-1},x',x')+bG(x_{n-1},x_n,x_n)].$$

Taking limit as $n \to +\infty$, we get

$$G(x',Tx',Tx') = 0,$$

which gives that $Tx' = x'$, that is, x' is a fixed point of T. Now if all elements of X are not fixed points of T, then there exists some $x^* \in X$ such that $Tx^* \neq x^*$.

On putting $x = y = z = x^*$ in (2.4.7), we have

$$0 \leq (b+c+d)\ G(x^*,Tx^*,Tx^*)$$

which implies that

$$b+c+d \geq 0.$$

Thus $a < 1$ as $a+b+c+d < 1$. Let y be another fixed point of T. Then

$$G(x',y,y) \;=\; G(Tx',Ty,Ty)$$

$$\leq\; aG(x',y,y)+bG(x',Tx',Tx')+(c+d)G(y,Ty,Ty)$$

$$=\; aG(x',y,y)$$

which implies that $G(x',y,y) = 0$ as $a < 1$, therefore, $x' = y$, that is, x' is a unique fixed point of T. Now, we furnish some examples which show that the hypothesis of Theorem 2.4.4 is satisfied, but not of Theorem 4.4.2 and 4.4.3.

Example Let $X = \{\alpha,\beta,\gamma,\delta\}$ and define a mapping $G: X\times\ X\times\ X \to\ [0,+\infty)$ as follows:

$G(x,x,x) = 0, \quad \text{for all} \quad x \in X,$

$G(\alpha,\beta,\beta) = G(\beta,\alpha,\beta) = G(\beta,\beta,\alpha) = G(\alpha,\alpha,\beta) = G(\alpha,\beta,\alpha) = G(\beta,\alpha,\alpha) = 2,$

$G(\alpha,\gamma,\gamma) = G(\gamma,\alpha,\gamma) = G(\gamma,\gamma,\alpha) = G(\alpha,\alpha,\gamma) = G(\alpha,\gamma,\alpha) = G(\gamma,\alpha,\alpha) = 1,$

$G(\alpha,\delta,\delta) = G(\delta,\alpha,\delta) = G(\delta,\delta,\alpha) = G(\alpha,\alpha,\delta) = G(\alpha,\delta,\alpha) = G(\delta,\alpha,\alpha) = 1,$

$G(\beta,\gamma,\gamma) = G(\gamma,\beta,\gamma) = G(\gamma,\gamma,\beta) = G(\beta,\beta,\gamma) = G(\beta,\gamma,\beta) = G(\gamma,\beta,\beta) = 2.1,$

$G(\beta,\delta,\delta) = G(\delta,\beta,\delta) = G(\delta,\delta,\beta) = G(\beta,\beta,\delta) = G(\beta,\delta,\beta) = G(\delta,\beta,\beta) = 1.3,$

$G(\gamma,\delta,\delta) = G(\delta,\gamma,\delta) = G(\delta,\delta,\gamma) = G(\gamma,\gamma,\delta) = G(\gamma,\delta,\gamma) = G(\delta,\gamma,\gamma) = 4.3,$

$G(\alpha,\delta,\gamma) = G(\alpha,\gamma,\delta) = G(\delta,\alpha,\gamma) = G(\delta,\gamma,\alpha) = G(\gamma,\alpha,\delta) = G(\gamma,\delta,\alpha) = 4.2,$

$G(x,y,z) = 5, \quad \text{otherwise}.$

It is easy to prove that (X, G) is a generalized G_b-complete metric space with constant $s = 2.5$. However, it is noticed that, with $x = \delta,\ y = \beta,\ z = \gamma,$

$$G(x,y,z) \nleq G(x,\beta,\beta) + G(\beta,y,z);$$

thus, G is not a G-metric on X. Also, with $x = \gamma,\ y = \delta,\ z = \alpha,$

$$G(x,y,y) = 4.3 \nleq 4.2 = G(x,y,z);$$

which means G is not a generalized b-metric on X.

Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \delta,\ T\gamma = \delta,\ T\delta = \alpha.$

Now for $a = 0.7,\ b = 0.07,\ c = 0.08,\ d = 0.09$, we have $a+b+c+d < 1,\ s(c+d) < 1$ and $a + b \geq 0.$

Also it is not hard to prove that (4.4.7) holds true. Here, α is the unique fixed point of T.

However, for $x = y = \alpha,\ z = \gamma,$ we notice that (4.4.2) does not hold true for any $k \in [0,1)$; and for $x = \alpha,\ y = z = \beta,$, (4.4.5) does not hold true for any $k \in [0, \frac{1}{5})$.

Example Let (X, G) be a generalized Gb-complete metric space as in Example 2.4.8.

Define a mapping $T : X \to X$ by $T\alpha = \alpha,\ T\beta = \beta,\ T\gamma = \gamma,\ T\delta = \delta.$

Then for $a = 1.4,\ b = -0.2,\ c = -3,\ d = -1$, we have $a+b+c+d < 1,\ s(c+d) < 1$

and $a + b \geq 0.$

In addition, it is evident that (4.4.7) is valid; yet, all of the components of X are found at fixed locations of T. In this section, we will illustrate several fixed point solutions for quasi-contraction type mappings within the context of generalised Gb-metric space.

Conclusion

The present paper examined fixed point theorems in the context of generalized Gb-metric spaces. The discussion shows that generalized Gb-complete metric spaces provide a powerful framework for extending classical fixed point results. The theorems considered in the study establish the existence and uniqueness of fixed points under different generalized contractive conditions. The paper also highlights that fixed points may exist even for discontinuous mappings, which makes the results stronger than many classical fixed point theorems. The examples support the validity and applicability of the theoretical results. Overall, the study contributes to the development of fixed point theory in generalized metric spaces and opens scope for further investigation of quasi-contraction type mappings and other generalized nonlinear mappings.

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