CONVERGENCE, STABILITY, AND APPLICATIONS OF FIXED POINT RESULTS IN GENERALIZED METRIC SPACES

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Abstract

Fixed point theory provides a mathematical language for proving that an iterative process has a stable limiting solution. In classical metric spaces, Banach's contraction principle links distance reduction, completeness, convergence, uniqueness, and computability. During the last two decades, however, the framework has expanded through b-metric spaces, G-metric spaces, cone metric spaces, partially ordered metric spaces, Jleli-Samet generalized metric spaces, F-contractions, multivalued contractions, and stability-oriented variants. This paper presents a conceptual and analytical study of convergence, stability, and applications of fixed point results in generalized metric spaces. It uses a qualitative doctrinal method supported by formula-based comparison, illustrative convergence graphs, and structured tables. The study argues that generalized metric settings are valuable when the ordinary triangle inequality, single-valued mapping assumption, or classical order-free distance model is too restrictive for nonlinear problems. At the same time, the paper emphasises that not every new distance structure creates a genuinely new theorem; some results can be metrized or reduced to older principles. The central finding is that a meaningful fixed point theorem must satisfy three criteria: an appropriate generalized distance structure, a contractive condition strong enough to force convergence, and a stability mechanism that protects the solution from computational and data perturbations. Applications are discussed in matrix equations, differential equations, integral equations, optimization, computational algorithms, and fractal-type iterative models.

Keywords: fixed point theory; generalized metric spaces; convergence; stability; b-metric spaces; G-metric spaces; cone metric spaces; F-contractions; Picard iteration; nonlinear analysis

1. Introduction

Fixed point theory studies conditions under which a mapping T has a point p such that T(p) = p. This apparently simple equation is foundational because many mathematical and applied problems can be reformulated as finding a point that remains unchanged under an operator. Existence of solutions of differential equations, equilibrium states, iterative numerical algorithms, matrix equations, optimization procedures, and integral equations often depend on converting the original problem into a fixed point problem. Once the fixed point exists, uniqueness and convergence results explain whether the theoretical solution can be approximated by repeated computation.

The classical model is the Banach contraction principle. It states that a contraction mapping on a complete metric space has a unique fixed point and that the Picard sequence converges to it. This theorem is powerful because it combines existence, uniqueness, constructive approximation, and error control in one statement. Yet many nonlinear problems are not naturally situated in ordinary metric spaces. Distances may be vector-valued, order-sensitive, multicomponent, asymmetric, b-metric type, or generated by a generalized topology. For this reason, modern fixed point research has moved from ordinary metric spaces toward generalized metric spaces.

Generalized metric spaces relax or reinterpret the distance axioms. A b-metric replaces the triangle inequality by a controlled inequality with a coefficient. A G-metric measures a three-point distance. A cone metric takes values in an ordered Banach space. A Jleli-Samet generalized metric allows a broad distance-like function that may not satisfy a classical triangle inequality but still supports convergence through suitable axioms. These structures were introduced to capture mathematical situations where the usual metric is too narrow or where the natural geometry of the problem is nonstandard (Mustafa & Sims, 2006; Huang & Zhang, 2007; Jleli & Samet, 2015).

The purpose of this paper is to analyse convergence, stability, and applications as three interdependent dimensions of fixed point theory in generalized metric spaces. Convergence explains why iterative sequences approach a fixed point.

Stability explains whether approximate sequences, numerical errors, perturbations, and residual errors preserve the same solution. Applications show why the abstract theory matters outside pure topology. The paper is not intended to prove a new theorem; rather, it provides an integrated research-style framework for comparing recent fixed point results and interpreting their relevance for applied nonlinear analysis.

The research question guiding the paper is: how do generalized metric structures extend fixed point theory while preserving convergence, stability, and applicability? This question is important because fixed point theory has become highly diverse. Some generalizations are mathematically deep, while others merely repackage classical metric results. Therefore, a critical study must not only describe theorems but also examine whether a generalized result provides genuine analytical value (Haghi et al., 2011).

2. Conceptual Background

2.1 Classical Metric Framework

A metric space is a pair (X, d), where X is a nonempty set and d is a distance function satisfying non-negativity, identity of indiscernibles, symmetry, and the triangle inequality. Completeness means every Cauchy sequence in X converges to an element of X. This property is essential because the Picard sequence created by iteration may be Cauchy before its limit is known. The contraction condition then transforms the topological concept of completeness into a constructive existence theorem.

$$d(Tx, Ty) \leq q\, d(x, y), \quad 0 \leq q < 1. \qquad (1)$$

Equation (1) is the classical contraction inequality. If T satisfies it on a complete metric space, then T has a unique fixed point p and $x_{n+1} = T x_n$ converges to p for any starting point x_0. The strength of this result is not only the existence of p but also the numerical method for approximating it.

$$x_{n+1} = T x_n, \quad n = 0,1,2, \ldots \qquad (2)$$

$$d(x_n, p) \leq q^n d(x_0,p) \quad \text{or} \quad d(x_n,p) \leq \left(q^n/(1-q)\right) d(x_1,x_0). \qquad (3)$$

2.2 Meaning of Generalized Metric Spaces

A generalized metric space is a distance-like environment in which the ordinary metric axioms are modified, extended, or embedded within a broader order or algebraic structure. The term does not refer to one single object. It includes families of spaces such as b-metric spaces, G-metric spaces, cone metric spaces, dislocated metric spaces, partial metric spaces, rectangular metric spaces, complex-valued metric spaces, and Jleli-Samet generalized metric spaces. These spaces differ in their treatment of triangle inequalities, self-distance, order structure, and convergence.

$$d(x,z) \leq s[d(x,y) + d(y,z)], \quad s \geq 1. \qquad (4)$$

Equation (4) describes a b-metric inequality. When s = 1, the usual triangle inequality is recovered. When s > 1, the space allows controlled weakening of triangularity. This is useful in settings where a natural distance behaves like a metric up to a scaling constant. Fixed point theorems in b-metric spaces therefore adjust contraction constants and convergence arguments to account for the coefficient s (Aydi et al., 2012).

In a cone metric space, the distance does not take real values but values in an ordered Banach space. This approach was introduced to enrich distance comparison by replacing the usual order on real numbers with a cone order. Such a model can represent multicomponent errors or vector-valued measurement, although later work has examined whether many cone metric results are reducible to classical metric ones (Huang & Zhang, 2007; Haghi et al., 2011).

Jleli-Samet generalized metric spaces take a broader route. They introduce an abstract distance structure that can recover many previously known spaces, including b-metric spaces and modular-type spaces. Their framework is significant because it shifts attention from merely modifying the triangle inequality to formulating convergence and completeness conditions that can support fixed point principles without requiring the ordinary metric structure (Jleli & Samet, 2015).

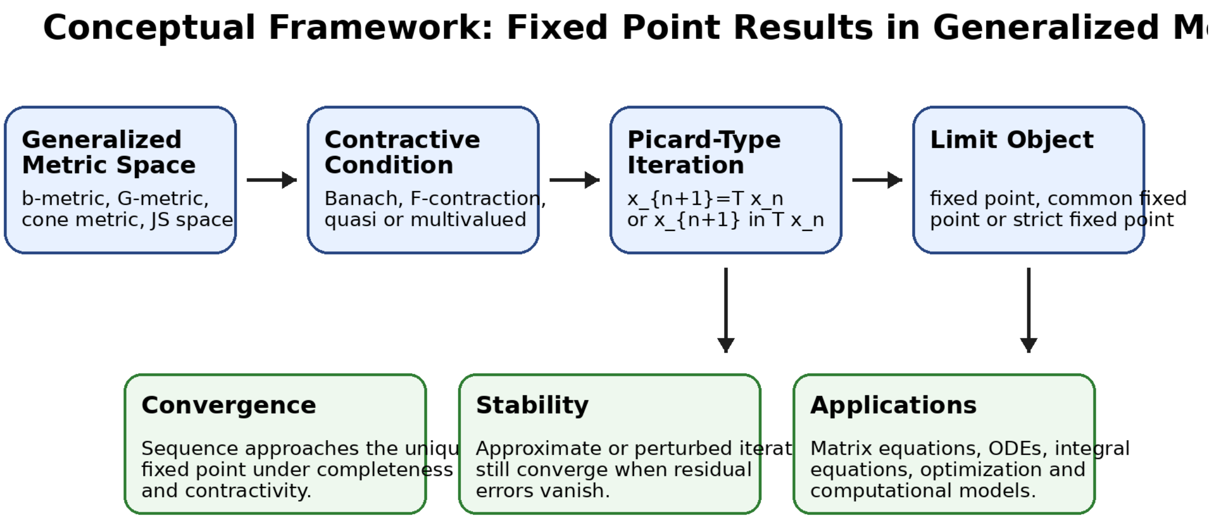

Figure 1. Conceptual framework linking generalized metric structure, contractive condition, iteration, convergence, stability and applications.

3. Review of Literature

Research on generalized metric fixed point theory has evolved through several streams. The first stream extends the underlying space. Mustafa and Sims (2006) proposed G-metric spaces to address weaknesses in earlier generalized metric models and to create a three-variable distance setting. Huang and Zhang (2007) introduced cone metric spaces and proved fixed point theorems for contractive mappings, thereby linking distance theory with ordered Banach spaces. Jleli and Samet (2015) later proposed a new generalized metric framework capable of recovering a wide range of earlier distance spaces.

The second stream extends the contraction condition. Wardowski (2012) introduced F-contractions, which generalize Banach-type contraction through an auxiliary function F. This approach is important because it weakens the form of the contraction while preserving the ability to force Cauchy behaviour. Aydi et al. (2012) extended fixed point analysis to set-valued quasi-contractions in b-metric spaces, showing that generalized spaces and multivalued mappings can interact fruitfully. Karapinar et al. (2016) developed fixed point theorems in new generalized metric spaces under broad contractive conditions, directly building on the Jleli-Samet framework.

The third stream examines order, applications, and reductions. Ran and Reurings (2004) applied ordered metric fixed point ideas to matrix equations, while Nieto and Rodríguez-López (2005) used ordered contractive mappings in ordinary differential equations. These works are relevant because generalized metric results often become valuable only when linked to concrete problems. At the same time, Haghi et al. (2011) warned that some fixed point generalizations are not real generalizations. Their critique is methodologically important: the value of a theorem should be judged by whether it provides new mathematical content, not merely new terminology.

The fourth stream concerns stability. Stability asks whether approximate or perturbed iterations still converge to the intended fixed point. Singh et al. (2012) emphasized round-off stability for multivalued maps and showed why computational fixed point theory must account for residual errors. Moga and Trușcă (2025) continued this stability-oriented line by proving stability properties for multivalued nonlinear graph contractions and applying them to integral inclusions. This recent development shows that fixed point theory is not merely about existence; it is also about robustness under imperfect data and computation.

Dumitrescu and Pitea (2022) further extended fixed point theorems in Jleli-Samet generalized metric spaces by studying almost contractions involving comparison functions. Their work is useful for interpreting convergence in settings where the contractive condition is not simply a linear inequality. Together these studies show that the modern field is organized around three pressures: relaxing the space, weakening the contraction, and preserving convergence/stability.

Table 1. Thematic synthesis of author-based literature used in the study.

Theme	Representative authors	Main contribution	Relevance to this paper
Space generalization	Mustafa & Sims (2006); Huang & Zhang (2007); Jleli & Samet (2015)	Introduced or expanded nonclassical distance structures such as G-metric, cone metric and generalized metric spaces.	Provides the conceptual base for analysing convergence beyond ordinary metric spaces.
Order and applications	Ran & Reurings (2004); Nieto & Rodríguez-López (2005)	Used ordered metric fixed point results to solve matrix equations and differential equations.	Shows how abstract fixed point principles can be applied to concrete nonlinear problems.
Contractive conditions	Wardowski (2012); Karapinar et al. (2016); Dumitrescu & Pitea (2022)	Developed F-contractions and generalized contraction frameworks in modern metric-type spaces.	Supports analysis of convergence under weak or nonlinear contraction assumptions.
Multivalued and b-metric results	Aydi et al. (2012)	Extended quasi-contraction fixed point results to set-valued mappings in b-metric spaces.	Links generalized distances with multivalued mappings and applications.
Critical validity	Haghi et al. (2011); Kadelburg & Radenović (2014)	Questioned reducibility and topological assumptions in generalized metric results.	Provides caution against treating every reformulation as a genuine generalization.
Stability	Singh et al. (2012); Moga & Trușcă (2025)	Studied round-off stability, residual error, and stability of multivalued nonlinear contractions.	Establishes the robustness dimension of fixed point theory.

4. Research Objectives

1. To explain the mathematical meaning of generalized metric spaces and their distinction from ordinary metric spaces.
2. To examine how fixed point theorems preserve convergence under b-metric, G-metric, cone metric and Jleli-Samet type frameworks.
3. To analyse the role of contraction conditions in forcing Cauchy behaviour and uniqueness of fixed points.
4. To evaluate stability concepts such as round-off stability, residual convergence and data dependence.
5. To compare the theoretical relevance of single-valued and multivalued fixed point results.

5. Research Methodology

This study adopts a qualitative, descriptive and analytical research design. It is qualitative because it interprets mathematical concepts, theorem structures, and applications rather than collecting empirical field data. It is descriptive because it presents the key types of generalized metric spaces and their associated fixed point results. It is analytical because it compares convergence, stability, and application value across different theorem families.

The study relies entirely on author-based academic sources published between 2002 and 2025. No ministry, state department, administrative report, or Wikipedia source is used. The selected references consist of journal articles in nonlinear analysis, fixed point theory, ordered metric spaces, b-metric spaces, cone metric spaces, generalized metric spaces, and stability theory. The selection criterion was relevance to convergence, stability, applications, and generalized metric structures.

The analytical method has three levels. First, the paper identifies the structure of the space, such as whether it is a b-metric, G-metric, cone metric, ordered metric or Jleli-Samet generalized metric space. Second, it examines the contraction form and determines how the contraction produces convergence. Third, it evaluates stability and application value by asking whether the fixed point result remains useful when computations, data or mappings are perturbed.

Table 2. Methodological framework of the study.

Element	Description	Material used	Purpose
Research type	Descriptive and analytical	Author-based journal literature from 2002-2025	To provide a structured account of the field
Core method	Conceptual and theorem-structure analysis	Definitions, contraction inequalities, stability criteria	To connect formulas with mathematical interpretation
Comparative basis	Convergence, stability and applications	Tables, formulas and illustrative graphs	To compare the usefulness of theorem families
Limitations	No new theorem is proved	Secondary mathematical sources only	To keep the paper interpretive rather than original proof-based

6. Generalized Metric Structures and Fixed Point Formulas

6.1 b-Metric Spaces

A b-metric space relaxes the triangle inequality by introducing a constant $s \geq 1$. This makes it possible to work with distances that grow faster than ordinary metric distances under composition. The coefficient s affects convergence proofs because each use of the generalized triangle inequality may amplify estimates. Therefore, contraction constants often need to be stronger than in ordinary metric spaces.

$$d(x,z) \leq s[d(x,y)+d(y,z)], \quad s \geq 1. \qquad (5)$$

For a Banach-type contraction in a b-metric space, the mapping T generally satisfies a condition of the form $d(Tx,Ty) \leq k\, d(x,y)$, but convergence depends on how k interacts with the b-metric coefficient. In practical terms, the larger the coefficient s, the more carefully the contraction must control error propagation. This explains why b-metric fixed point proofs often contain additional technical restrictions compared with standard metric results.

6.2 G-Metric and Generalized Distance Structures

G-metric spaces use a function $G(x,y,z)$ involving three points rather than a two-point distance. The intention is to describe the geometry of convergence through triples. A sequence may be analysed through conditions such as $G(x_n, x_m, x_m)$ becoming small. Mustafa and Sims (2006) developed this approach to correct weaknesses in earlier generalized metric theory and to create a more coherent structure for fixed point results.

$$G: X \times X \times X \to [0,\infty), \quad \text{with convergence controlled by } G(x_n, x_m, x_m). \qquad (6)$$

6.3 Cone Metric Spaces

In a cone metric space, the distance between two points is an element of an ordered Banach space rather than a nonnegative real number. The inequality $d(x,y) \leq d(x,z)+d(z,y)$ is interpreted through an order induced by a cone. This framework can model vector-valued error and multicomponent comparison. However, later critiques show that some cone metric results may be equivalent to classical metric results under metrizability, which means that their novelty must be assessed carefully.

$$d: X \times X \to E, \quad \text{where } E \text{ is an ordered Banach space and } 0_E \leq d(x,y). \qquad (7)$$

6.4 Jleli-Samet Generalized Metric Spaces

The Jleli-Samet framework is important because it does not merely alter a metric axiom; it introduces a generalized distance concept that can recover many well-known spaces. Its significance lies in providing a unified language for ordinary metrics, b-metrics, dislocated metrics and modular-type structures. Fixed point theorems in this environment focus on whether convergence and completeness can be defined in a way that is strong enough to support contractive arguments.

$$D(x,y) = 0 \Rightarrow x = y, \quad \text{and convergence is governed by } D\text{-limits rather than an ordinary metric.} \qquad (8)$$

Table 3. Comparison of generalized metric structures.

Space type	**Distance object**	**Key modification**	**Convergence issue**	**Typical theorem role**
Metric space	d(x,y) in R+	Ordinary triangle inequality	Cauchy sequence and complete space	Baseline for Banach contraction
b-metric space	d(x,y) in R+	Triangle inequality has coefficient $s \geq 1$	Contraction must control coefficient s	Useful for controlled non-metric distances
G-metric space	G(x,y,z)	Three-point distance function	Convergence expressed through triples	Useful for generalized topological distance
Cone metric space	d(x,y) in ordered Banach space	Distance is vector/order-valued	May be reducible to metric form	Useful for vector-valued error interpretation
Jleli-Samet space	Generalized D(x,y)	Broad nonclassical distance	Needs abstract completeness condition	Unifies multiple generalized spaces
Multivalued setting	Hausdorff-type distance H(A,B)	Mapping takes values in subsets	Fixed point becomes p in T(p)	Useful for inclusions and control models

7. Convergence Analysis

Convergence is the central bridge between fixed point existence and practical computation. A theorem that only states that a fixed point exists is less useful if it does not explain how to approximate it. In the classical setting, the Picard sequence $x_{n+1}=Tx_n$ converges geometrically when T is a contraction. The same logic is adapted in generalized metric spaces, but the proof must account for the modified distance structure.

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq q\, d(x_n, x_{n-1}). \qquad (9)$$

Repeated application of equation (9) gives a geometric decrease of successive increments. This is the basic engine of convergence proofs. If the space is complete, the resulting Cauchy sequence converges to a limit p. A separate argument then shows that p is a fixed point, usually by continuity, contractivity, orbital completeness, or a limiting inequality.

In generalized metric spaces, convergence must be treated with more caution. For example, in b-metric spaces, repeated use of the relaxed triangle inequality may introduce powers of the coefficient s. In cone metric spaces, convergence must be interpreted through the cone order. In multivalued settings, convergence may concern distance from a point to a set or convergence of a sequence of sets under a Hausdorff-type metric. These differences do not remove the core idea: contractive behaviour must dominate error propagation.

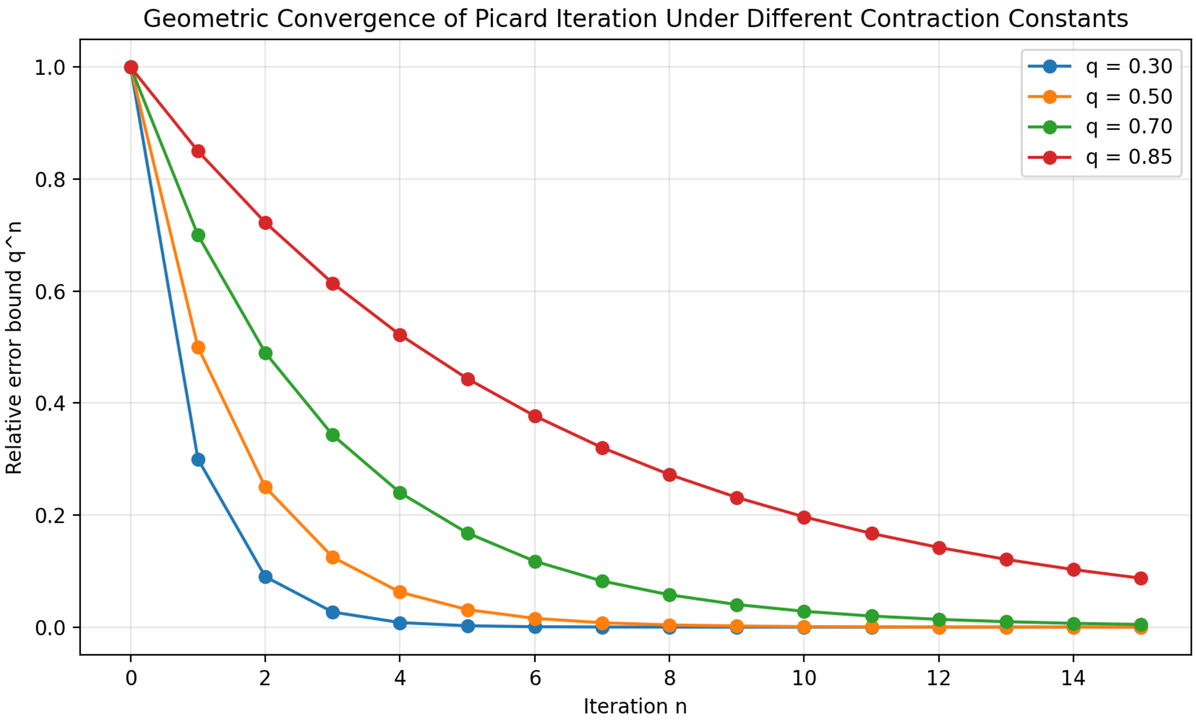

Figure 2. Illustrative geometric convergence rates for Picard iteration under different contraction constants.

Figure 2 illustrates the theoretical error pattern q^n. Lower values of q produce faster convergence. The curve is not based on empirical data; it visualizes the standard error structure of contraction-based iteration. In applications, the practical value of a fixed point theorem increases when it gives an explicit or estimable convergence rate, because numerical algorithms require stopping criteria and error bounds.

$$\text{If } 0 \le q < 1, \text{ then } q^n \to 0 \text{ as } n \to \infty. \qquad (10)$$

Equation (10) is mathematically elementary but conceptually decisive. It is the reason a contraction can turn local distance reduction into global convergence. Generalized fixed point theorems can be understood as attempts to preserve this mechanism while relaxing the space, the mapping, or the contraction form.

8. Stability Analysis

Stability asks whether the convergence result survives errors. In exact theory, the sequence $x_{n+1}=Tx_n$ is computed perfectly. In practice, numerical iteration may produce y_{n+1} close to $T y_n$ but not equal to it. Data may be perturbed, the operator may be approximated, or round-off errors may accumulate. A stable fixed point result ensures that such imperfections do not destroy convergence.

$$\varepsilon_n = d(y_{n+1}, T y_n). \qquad (11)$$

The quantity ε_n in equation (11) is a residual error. If ε_n tends to zero and the contraction structure is strong enough, the approximate sequence $\{y_n\}$ may still converge to the same fixed point p. This is the essence of T-stability and round-off stability. Singh et al. (2012) show that stability of iterative procedures is important in numerical computation, fractal generation, computer programming and applied analysis.

$$\varepsilon_n \to 0 \;\Rightarrow\; y_n \to p. \qquad (12)$$

Equation (12) represents a typical stability implication. It does not hold automatically; it depends on completeness, contractivity, and suitable assumptions on the map. In multivalued settings, the residual may be written through a Hausdorff-type distance, such as $H(y_{n+1}, T y_n)$, where $T y_n$ is a set. This makes stability more complex but also more relevant to differential inclusions and control problems.

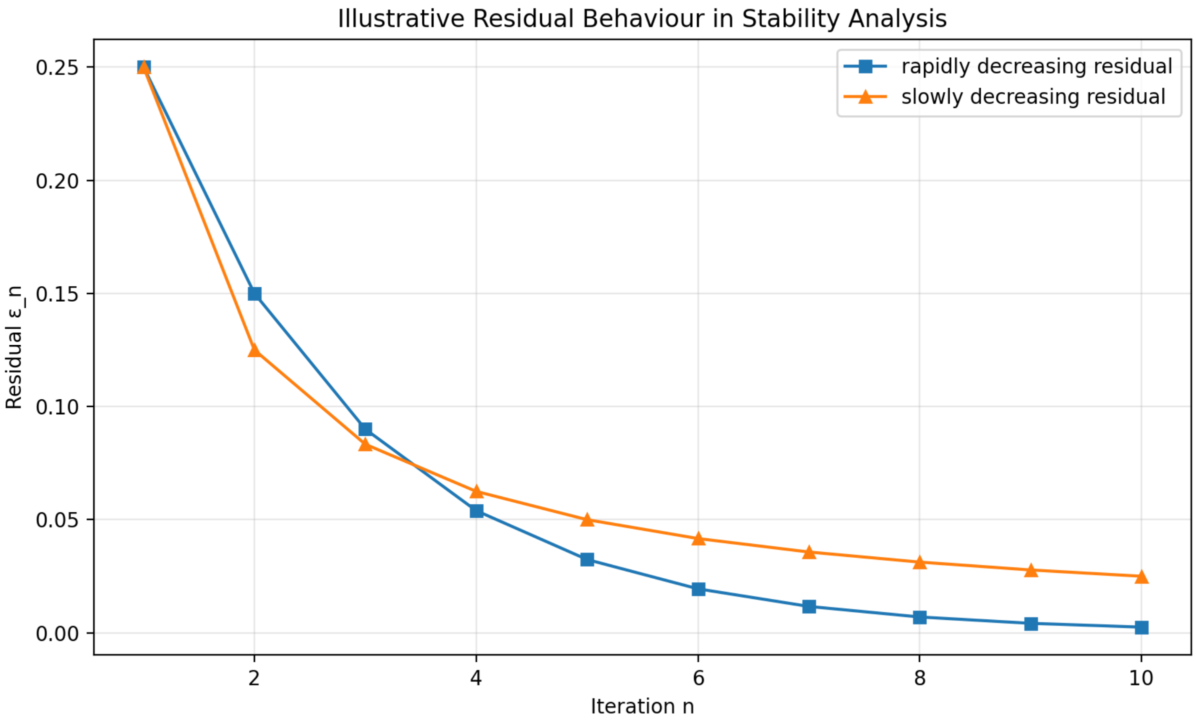

Figure 3. Illustrative residual sequences used to interpret stability of approximate iterations.

Figure 3 compares two residual patterns. Both residual sequences tend to zero, but one does so more rapidly. In a stable theorem, the decay of residual error supports convergence of the approximate orbit. In applied computation, residual monitoring is often more practical than direct knowledge of the fixed point because the exact solution is unknown.

8.1 Data Dependence and Ulam-Hyers Stability

Data dependence concerns how the fixed point changes when the operator changes. If T and S are nearby mappings, one asks whether their fixed points p_T and p_S are also close. This issue is essential in models where parameters are measured approximately or estimated from data. Stability in the Ulam-Hyers sense asks whether an approximate solution of x = Tx is close to an exact fixed point.

$$d(x,Tx) \leq \varepsilon \;\Rightarrow\; d(x,p) \leq C\varepsilon. \qquad (13)$$

Equation (13) expresses a typical Ulam-Hyers stability idea. It converts approximate satisfaction of the fixed point equation into a distance estimate from the true fixed point. Moga and Truşcă (2025) demonstrate how modern multivalued graph contraction theory includes stability, localization, and applications to integral inclusions, showing the increasing importance of this perspective.

Table 4. Stability concepts in fixed point theory.

Stability concept	**Basic idea**	**Mathematical indicator**	**Application relevance**
Picard stability	Approximate iteration converges to same fixed point	ε_n = d(y_{n+1},Ty_n) → 0	Numerical iteration and algorithms
Round-off stability	Finite precision errors do not destroy convergence	Accumulated residuals remain controlled	Computer computation and fractal generation
Data dependence	Fixed point set changes continuously with operator	T close to S implies p_T close to p_S	Parameter-sensitive models

Stability concept	Basic idea	Mathematical indicator	Application relevance
Ulam-Hyers stability	Approximate solution lies near exact solution	$d(x,Tx) \le \varepsilon$ implies $d(x,p) \le C\varepsilon$	Approximate modelling and perturbation theory
Set-valued stability	Selections and set images preserve convergence	Hausdorff-type residuals converge	Control, inclusions and economic models

9. Applications of Fixed Point Results

9.1 Matrix Equations

Matrix equations can often be rewritten as operator equations on ordered or metric spaces. Ran and Reurings (2004) showed how fixed point results in partially ordered sets can be used for linear and nonlinear matrix equations. This application is significant because it demonstrates that fixed point theory is not restricted to abstract topology. When matrix equations contain monotone structure, an ordered metric approach can provide existence and uniqueness results.

$$X = Q + A^{*} F(X) A. \qquad (14)$$

Equation (14) represents a typical fixed point reformulation of a matrix equation. The solution X is a fixed point of the operator $\Phi(X)=Q+A^{*}F(X)A$. If Φ is contractive or monotone contractive under an appropriate matrix order and distance, then fixed point theory can establish solvability and provide an iterative scheme.

9.2 Differential and Integral Equations

Differential equations and integral equations are among the most common applications of fixed point theory. A boundary value or initial value problem can be converted into an integral equation. The integral operator is then studied as a mapping on a function space. If it is contractive in a suitable distance, the fixed point is the desired solution. Nieto and Rodríguez-López (2005) provide a prominent example of applying ordered fixed point theorems to ordinary differential equations.

$$x(t) = g(t) + \int_a^b K(t,s,x(s))\, ds. \qquad (15)$$

Equation (15) gives the standard operator form of an integral equation. The right-hand side defines a mapping T on a space of functions. A fixed point $x = Tx$ solves the equation. Generalized metric spaces become useful when the natural function distance is weighted, ordered, vector-valued, or not perfectly metric in the classical sense.

9.3 Optimization and Computational Algorithms

Optimization algorithms often generate sequences intended to approach a stationary point or minimizer. Although not every optimization process is a contraction, many convergence analyses use fixed point language. For example, projection methods, proximal iterations, and iterative schemes can be studied by identifying a map whose fixed points correspond to solutions. Stability matters here because algorithms operate with finite precision and noisy data.

9.4 Fractal and Multivalued Models

Multivalued fixed point theory is relevant where a state may evolve into a set of possible states. This occurs in differential inclusions, control theory, game theory and fractal generation. A multivalued map T has a fixed point when $p \in T(p)$, not necessarily when p equals a single image. Stability of such maps is important because small errors in set-valued iteration can produce large changes in approximate attractors if the process is not stable.

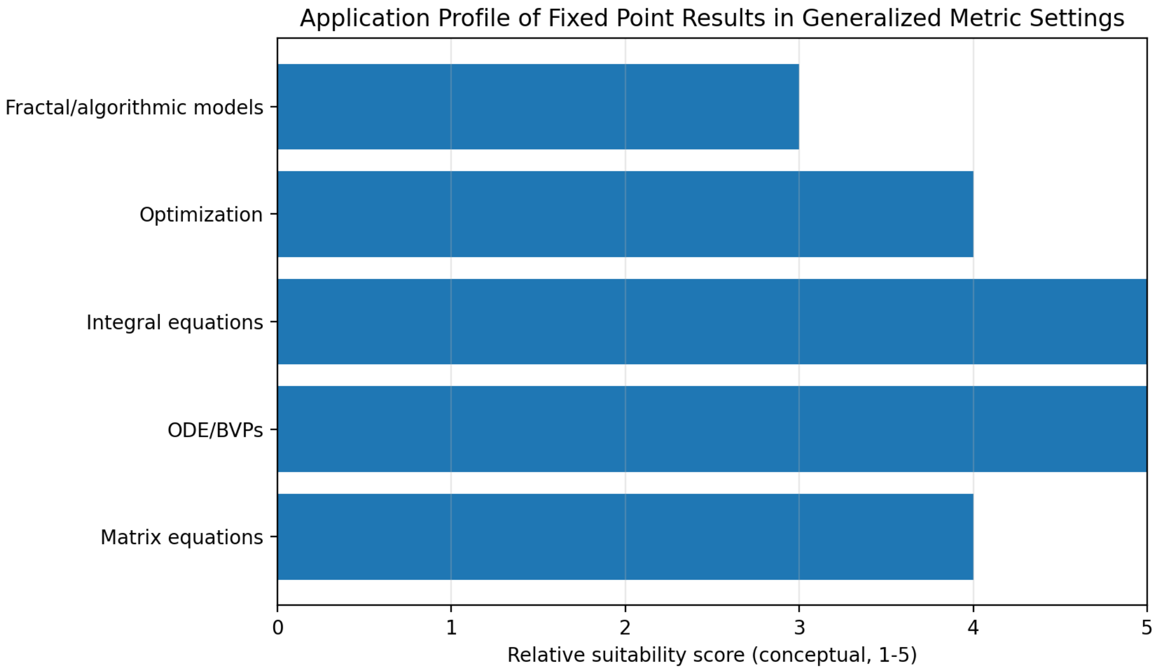

Figure 4. Conceptual application profile of fixed point results in generalized metric settings.

Table 5. Application matrix for generalized fixed point results.

Application area	**Fixed point model**	**Useful generalized structure**	**Reason for relevance**
Matrix equations	$X = \Phi(X)$	Ordered metric spaces	Matrix order and monotonicity can support convergence
Ordinary differential equations	x = Tx in a function space	Ordered or weighted metric spaces	Lower-upper solution methods use monotone contraction
Integral equations	$x(t)=g(t)+\int K(t,s,x(s))ds$	Generalized metric or cone metric spaces	Kernel estimates may be vector-valued or weighted
Differential inclusions	$p \in T(p)$	Multivalued generalized metric spaces	Solution sets are naturally set-valued
Optimization algorithms	x_{n+1}=T x_n	Metric, b-metric or F-contraction spaces	Iterative maps need convergence and residual control
Fractal models	Set iteration under multivalued maps	Hausdorff-type metric spaces	Stable iteration preserves attractor generation

10. Findings and Discussion

The first finding is that generalized metric spaces are valuable only when they solve a genuine modelling problem. A b-metric is useful when the natural distance obeys a weakened triangle inequality. A cone metric is useful when distance has vector or ordered components. A multivalued framework is useful when the operator returns a set. However, if a generalized theorem can be completely reduced to a classical metric theorem without adding insight, its value is mainly notational rather than mathematical.

The second finding is that convergence remains the core requirement. No matter how general the distance space is, a fixed point theorem must still produce a Cauchy-type sequence and identify its limit as a fixed point. This means that generalized fixed point theory is not an abandonment of classical convergence logic. It is a reconstruction of that logic in a wider environment.

The third finding is that stability is essential for applications. Many theoretical papers prove existence and uniqueness, but applied modelling requires robustness. In computation, one rarely obtains exact iteration. Residual errors, rounding errors, discretization and data perturbation are unavoidable. Stability theory explains when these imperfections remain harmless.

The fourth finding is that multivalued fixed point theory is especially important for applications involving uncertainty. Differential inclusions, control systems, set-valued optimization and economic models often produce a family of possible outcomes rather than a single next state. Generalized metric spaces provide a flexible environment for such problems, but the use of Hausdorff-type distances introduces technical difficulties.

The fifth finding is that modern generalized metric fixed point theory is moving from simple theorem extension toward integrated convergence-stability-application frameworks. Recent works such as Moga and Trușcă (2025) are notable because they combine fixed point existence, approximation, stability and application to integral inclusions rather than treating existence alone as the final result.

11. Critical Evaluation

A critical issue in this field is the distinction between genuine generalization and formal generalization. A formal generalization changes terminology or the surrounding structure but does not produce a theorem that is essentially different from the classical result. A genuine generalization either applies to spaces that cannot be handled by ordinary metric methods, weakens a contraction in a meaningful way, improves convergence or stability information, or enables new applications. Haghi et al. (2011) are important because they remind researchers that novelty claims should be tested against reducibility.

Another issue is the balance between abstraction and usability. Highly abstract spaces may produce elegant theorems but can become difficult to apply. Conversely, application-driven spaces may be less elegant but more useful for differential equations, matrix equations and computational problems. A strong research paper in this field should therefore explain both the theorem and its modelling motivation.

A final issue concerns proof economy. Many fixed point results use similar proof templates: construct an orbit, show it is Cauchy, use completeness, then verify that the limit is a fixed point. Generalized theories become valuable when they modify at least one step in this proof architecture in a necessary and nontrivial way. Otherwise, the theorem risks becoming a restatement of a known principle.

12. Recommendations for Future Research

- Future studies should combine existence, convergence rate, stability and application within the same theorem framework rather than proving existence alone.
- Researchers should state explicitly whether a generalized metric theorem is reducible to an ordinary metric theorem and what advantage the generalized structure provides.
- More work is needed on computable error bounds in b-metric and Jleli-Samet type spaces, especially where the contraction constant interacts with a generalized triangle coefficient.
- Applications to differential inclusions, optimization algorithms and data-driven models should include residual-based stability conditions because exact computation is rarely possible.
- Studies should distinguish between single-valued, multivalued and graph-based contractions, since each requires different convergence and stability arguments.
- Future research should develop examples that are not artificially constructed but arise from real functional equations, matrix problems or numerical algorithms.

13. Conclusion

This paper examined convergence, stability and applications of fixed point results in generalized metric spaces. The analysis shows that generalized fixed point theory is best understood as an effort to preserve the power of Banach's contraction principle while widening the type of distance, mapping and application environment. Generalized metric spaces such as b-metric spaces, G-metric spaces, cone metric spaces and Jleli-Samet spaces allow fixed point methods to operate in settings where ordinary metric assumptions are too restrictive. However, such generalization is valuable only when it gives new mathematical or applied insight.

Convergence remains the core of the theory. Whether the space is classical or generalized, a meaningful theorem must show that an iterative sequence approaches a fixed point. Stability extends this by asking whether approximate iterations, residual errors and data perturbations preserve convergence. Applications in matrix equations, ordinary differential equations, integral equations, optimization and multivalued models demonstrate why stability is not optional. A fixed point theorem becomes practically powerful when it can prove existence, describe approximation, control error and survive perturbation.

The overall conclusion is that fixed point results in generalized metric spaces should be evaluated through three questions: Does the generalized space model a real mathematical need? Does the contractive condition genuinely force convergence? Does the theorem provide stability or application value beyond a formal restatement of known principles? When all three answers are positive, generalized metric fixed point theory becomes a significant tool for modern nonlinear analysis and computational mathematics.

References

Aydi, H., Bota, M.-F., Karapınar, E., & Mitrović, S. (2012). A fixed point theorem for set-valued quasi-contractions in b-metric spaces. Fixed Point Theory and Applications, 2012, Article 88. https://doi.org/10.1186/1687-1812-2012-88

Dumitrescu, D., & Pitea, A. (2022). Fixed point theorems on almost (φ, θ)-contractions in Jleli-Samet generalized metric spaces. Mathematics, 10(22), Article 4239. https://doi.org/10.3390/math10224239

Haghi, R. H., Rezapour, S., & Shahzad, N. (2011). Some fixed point generalizations are not real generalizations. Nonlinear Analysis: Theory, Methods & Applications, 74(5), 1799–1803. https://doi.org/10.1016/j.na.2010.10.052

Huang, L.-G., & Zhang, X. (2007). Cone metric spaces and fixed point theorems of contractive mappings. Journal of Mathematical Analysis and Applications, 332(2), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087

Jleli, M., & Samet, B. (2015). A generalized metric space and related fixed point theorems. Fixed Point Theory and Applications, 2015, Article 61. https://doi.org/10.1186/s13663-015-0312-7

Kadelburg, Z., & Radenović, S. (2014). Fixed point results in generalized metric spaces without Hausdorff property. Mathematical Sciences, 8, Article 125. https://doi.org/10.1007/s40096-014-0125-6

Karapınar, E., O'Regan, D., Roldán López de Hierro, A. F., & Shahzad, N. (2016). Fixed point theorems in new generalized metric spaces. Journal of Fixed Point Theory and Applications, 18, 645–671. https://doi.org/10.1007/s11784-016-0301-4

Moga, M., & Truşcă, R. (2025). Fixed point and stability results for multi-valued nonlinear graph contractions in complete metric spaces. The Journal of Analysis, 33, 717–742. https://doi.org/10.1007/s41478-024-00858-6

Mustafa, Z., & Sims, B. (2006). A new approach to generalized metric spaces. Journal of Nonlinear and Convex Analysis, 7(2), 289–297.

Nieto, J. J., & Rodríguez-López, R. (2005). Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order, 22, 223–239. https://doi.org/10.1007/s11083-005-9018-5

Ran, A. C. M., & Reurings, M. C. B. (2004). A fixed point theorem in partially ordered sets and some applications to matrix equations. Proceedings of the American Mathematical Society, 132(5), 1435–1443. https://doi.org/10.1090/S0002-9939-03-07220-4

Singh, S. L., Mishra, S. N., & Jain, S. (2012). Round-off stability for multi-valued maps. Fixed Point Theory and Applications, 2012, Article 12. https://doi.org/10.1186/1687-1812-2012-12

Wardowski, D. (2012). Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory and Applications, 2012, Article 94. https://doi.org/10.1186/1687-1812-2012-94

Appendix A: Key Formula Summary

Table 6. Formula summary used in the paper.

Concept	**Formula**
Banach contraction	$d(Tx,Ty) \leq qd(x,y),\ 0 \leq q < 1$
Picard iteration	$x_{n+1}=Tx_n$
b-metric inequality	$d(x,z) \leq s[d(x,y)+d(y,z)],\ s \geq 1$
Residual error	$\varepsilon_n=d(y_{n+1},Ty_n)$
Stability implication	$\varepsilon_n \to 0 \Rightarrow y_n \to p$
Ulam-Hyers type estimate	$d(x,Tx) \leq \varepsilon \Rightarrow d(x,p) \leq C\varepsilon$
Integral operator form	$x(t)=g(t)+\int_a^b K(t,s,x(s))ds$