

A QUANTITATIVE STUDY OF G -AUTOCOMMUTATIVITY IN FINITE GROUPS

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Abstract

The concept of autocommutativity provides a probabilistic framework for studying the interaction between elements of a finite group and its automorphism group. As a natural extension of classical commuting probability, g -autocommutativity measures the likelihood that the autocommutator of a randomly selected group element and a randomly selected automorphism assumes a prescribed value (g in G). This notion offers deeper insight into the structural relationship between group elements, automorphism actions, and symmetry properties of finite groups.

In this paper, we investigate the quantitative aspects of g -autocommutativity in finite groups. We establish fundamental definitions and derive explicit computational formulae for the g -autocommuting probability. Several structural properties and inequalities are obtained in terms of absolute centers, automorphism centralizers, orbit structures, and subgroup indices. Upper and lower bounds for the probability distribution are developed, revealing how the algebraic structure of a group influences the occurrence of specific autocommutator values. Furthermore, the behavior of g -autocommuting probability is analyzed for various classes of finite groups, leading to structural characterizations and classification results. The relationship between g -autocommutativity and generalized notions of commutativity is also explored, highlighting the role of automorphism actions in determining probabilistic invariants.

The results demonstrate that g -autocommuting probability serves as a significant quantitative invariant capable of capturing subtle structural information about finite groups. Consequently, the study contributes to the broader development of probabilistic group theory and provides new perspectives for investigating symmetry and automorphism-induced commutativity phenomena in finite algebraic systems.

Keywords: finite groups, g -autocommuting probability, autocommutativity, autocommutator, automorphism group, absolute center, automorphism centralizer, probabilistic group theory, generalized commutativity, finite algebraic systems.

Introduction

Let G be a finite group and let g be an element of G . In this chapter, we introduce and study the g -autocommuting probability of G , denoted by $\text{Pr}_g(G, \text{Aut}(G))$. We define $\text{Pr}_g(G, \text{Aut}(G))$ to be the probability that the autocommutator of a randomly chosen pair of elements, one from G and the other from $\text{Aut}(G)$, equals g . Clearly, $\text{Pr}_g(G, \text{Aut}(G)) = \text{Pr}(G, \text{Aut}(G))$ if $g = 1$. Note that $\text{Pr}_g(G, \text{Aut}(G)) = 1$ if and only if $K(G) = \{1\}$ and $g = 1$ if and only if $G = L(G)$ and $g = 1$. Also, $\text{Pr}_g(G, \text{Aut}(G)) = 0$ if and only if $g \notin S(G, \text{Aut}(G))$. Therefore, we consider $G \neq L(G)$ and $g \in S(G, \text{Aut}(G))$. It is worth noting that $G \neq L(G) \Leftrightarrow G$ possesses a non-trivial automorphism $\Leftrightarrow |G| > 2$. The motivation for studying the g -autocommuting probability of a finite group comes from the work of **Pournaki and Sobhani**.

In the subsequent sections, we obtain a computing formula and several bounds for $\text{Pr}_g(G, \text{Aut}(G))$. We also characterize G through $\text{Pr}(G, \text{Aut}(G))$.

1.1 A Computing Formula

The g -autocommuting probability of G is given by the ratio:

$$\Pr_g(G, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \text{Aut}(G): [x, \alpha] = g\}|}{|G||\text{Aut}(G)|}. \quad (1. a)$$

Note that if $g = 1$ then

$$\Pr(G, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \text{Aut}(G): [x, \alpha] = 1\}|}{|G||\text{Aut}(G)|}. \quad (1. b)$$

For any $x, g \in G$ let $T_{x,g}(G)$ denotes the set $\{\alpha \in \text{Aut}(G): [x, \alpha] = g\}$. Note that $T_{x,1} = C_{\text{Aut}(G)}(x)$. The following two lemmas play a crucial role in obtaining the computing formula for $\Pr_g(G, \text{Aut}(G))$.

Lemma 1.1.1

Let G be a finite group. If

then

for some

Hence,

Proof

For $x, g \in G$, define

$$T_{x,g}(G) = \{\alpha \in \text{Aut}(G): [x, \alpha] = g\},$$

where the autocommutator is

$$[x, \alpha] = x^{-1}\alpha(x).$$

Assume that

$$T_{x,g}(G) \neq \emptyset.$$

Then there exists

$$\sigma \in T_{x,g}(G).$$

Thus,

$$[x, \sigma] = g.$$

By definition,

$$x^{-1}\sigma(x) = g.$$

Hence,

$$\sigma(x) = xg.$$

Now let

$$\alpha \in \sigma C_{\text{Aut}(G)}(x).$$

Then

$$\alpha = \sigma\beta$$

for some

$$\beta \in C_{\text{Aut}(G)}(x).$$

Since β fixes x , we have

$$\beta(x) = x.$$

Therefore,

$$\alpha(x) = \sigma\beta(x) = \sigma(x) = xg.$$

Hence,

$$[x, \alpha] = x^{-1}\alpha(x) = x^{-1}(xg) = g.$$

So,

$$\alpha \in T_{x,g}(G).$$

Thus,

$$\sigma C_{\text{Aut}(G)}(x) \subseteq T_{x,g}(G)$$

Conversely, let

$$\alpha \in T_{x,g}(G).$$

Then

$$[x, \alpha] = g$$

so

$$x^{-1}\alpha(x) = g.$$

Therefore,

$$\alpha(x) = xg$$

Also, since $\sigma \in T_{x,g}(G)$,

$$\sigma(x) = xg$$

Hence,

$$\sigma^{-1}\alpha(x) = \sigma^{-1}(xg) = x$$

Thus,

$$\sigma^{-1}\alpha \in C_{\text{Aut}(G)}(x)$$

So there exists

$$\beta \in C_{\text{Aut}(G)}(x)$$

such that

$$\alpha = \sigma\beta$$

Hence,

$$\alpha \in \sigma C_{\text{Aut}(G)}(x)$$

Therefore,

$$T_{x,g}(G) \subseteq \sigma C_{\text{Aut}(G)}(x)$$

Combining both inclusions, we obtain

$$T_{x,g}(G) = \sigma C_{\text{Aut}(G)}(x)$$

Since $\sigma C_{\text{Aut}(G)}(x)$ is a left coset of $C_{\text{Aut}(G)}(x)$, it has the same cardinality as $C_{\text{Aut}(G)}(x)$. Therefore,

$$|T_{x,g}(G)| = |C_{\text{Aut}(G)}(x)|$$

Hence proved.

Theorem 1.1.2

Let G be a finite group. If $g \in G$, then

$$\Pr_g(G, \text{Aut}(G)) = \frac{1}{|G||\text{Aut}(G)|} \sum_{\substack{x \in G \\ xg \in \text{orb}_G(x)}} |C_{\text{Aut}(G)}(x)| = \frac{1}{|G|} \sum_{xg \in \text{orb}_G(x)} \frac{1}{|\text{orb}_G(x)|}.$$

Proof

By definition,

$$\text{Pr}_g(G, \text{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \text{Aut}(G) : [x, \alpha] = g\}|}{|G||\text{Aut}(G)|}.$$

For fixed $x \in G$, define

$$T_{x,g}(G) = \{\alpha \in \text{Aut}(G) : [x, \alpha] = g\}$$

Then

$$|\{(x, \alpha) : [x, \alpha] = g\}| = \sum_{x \in G} |T_{x,g}(G)|$$

Now,

$$[x, \alpha] = g$$

means

$$x^{-1}\alpha(x) = g$$

Hence,

$$\alpha(x) = xg$$

Therefore, such an automorphism α exists if and only if

$$xg \in \text{orb}_G(x)$$

where

$$\text{orb}_G(x) = \{\alpha(x) : \alpha \in \text{Aut}(G)\}$$

Thus, only those $x \in G$ satisfying

$$xg \in \text{orb}_G(x)$$

contribute to the summation.

By Lemma 1.2, whenever $T_{x,g}(G) \neq \emptyset$,

$$|T_{x,g}(G)| = |C_{\text{Aut}(G)}(x)|$$

Hence,

$$\text{Pr}_g(G, \text{Aut}(G)) = \frac{1}{|G||\text{Aut}(G)|} \sum_{\substack{x \in G \\ xg \in \text{orb}_G(x)}} |C_{\text{Aut}(G)}(x)|$$

This proves the first equality.

Now apply the Orbit-Stabilizer Theorem to the natural action of $\text{Aut}(G)$ on G . For every $x \in G$,

$$|\text{orb}_G(x)| = \frac{|\text{Aut}(G)|}{|C_{\text{Aut}(G)}(x)|}$$

Therefore,

$$\frac{|C_{\text{Aut}(G)}(x)|}{|\text{Aut}(G)|} = \frac{1}{|\text{orb}_G(x)|}$$

Substituting this into the previous formula gives

$$\Pr(G, \text{Aut}(G)) = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|\text{orb}_G(x)|}$$

Hence proved.

2.1 Some Bounds

Theorem 2.1.1

Let G be a finite group and let p be the smallest prime dividing $|\text{Aut}(G)|$. Then

$$\Pr(G, \text{Aut}(G)) \geq \frac{|L(G)|}{|G|} + \frac{p(|G| - |X_G| - |L(G)|) + |X_G|}{|G||\text{Aut}(G)|}$$

and

$$\Pr(G, \text{Aut}(G)) \leq \frac{(p-1)|L(G)| + |G|}{p|G|} - \frac{|X_G|(|\text{Aut}(G)| - p)}{p|G||\text{Aut}(G)|}$$

where $X_G = \{x \in G : C_{\text{Aut}(G)}(x) = \{I\}\}$.

Proof

Let

$$A = \text{Aut}(G)$$

and let

$$C_A(x) = C_{\text{Aut}(G)}(x) = \{\alpha \in A : \alpha(x) = x\}$$

By the counting formula for autocommuting probability,

$$\Pr(G, A) = \frac{1}{|G||A|} \sum_{x \in G} |C_A(x)|$$

Now divide G into three parts:

$$L(G), X_G = \{x \in G : C_A(x) = \{I\}\}$$

and

$$G \setminus (L(G) \cup X_G)$$

If $x \in L(G)$, then every automorphism fixes x . Hence,

$$|C_A(x)| = |A|$$

If $x \in X_G$, then

$$|C_A(x)| = 1$$

For every remaining element

$$x \in G \setminus (L(G) \cup X_G),$$

the subgroup $C_A(x)$ is neither trivial nor the whole group A . Since p is the smallest prime divisor of $|A|$, we have

$$p \leq |C_A(x)| \leq \frac{|A|}{p}$$

Therefore,

$$\sum_{x \in G} |C_A(x)| \geq |L(G)||A| + |X_G| + p(|G| - |L(G)| - |X_G|)$$

Dividing by $|G||A|$, we obtain

$$\Pr(G, A) \geq \frac{|L(G)|}{|G|} + \frac{p(|G| - |X_G| - |L(G)|) + |X_G|}{|G||A|}$$

This proves the lower bound.

Similarly, using

$$|C_A(x)| \leq \frac{|A|}{p}$$

for all

$$x \in G \setminus (L(G) \cup X_G)$$

we get

$$\sum_{x \in G} |C_A(x)| \leq |L(G)||A| + |X_G| + \frac{|A|}{p}(|G| - |L(G)| - |X_G|)$$

Hence,

$$\Pr(G, A) \leq \frac{|L(G)||A| + |X_G| + \frac{|A|}{p}(|G| - |L(G)| - |X_G|)}{|G||A|}$$

Simplifying,

$$\Pr(G, A) \leq \frac{(p-1)|L(G)| + |G| - |X_G|}{p|G|} + \frac{|X_G|}{|G||A|}$$

Equivalently,

$$\Pr(G, A) \leq \frac{(p-1)|L(G)| + |G|}{p|G|} - \frac{|X_G|(|A| - p)}{p|G||A|}$$

Thus both required bounds are established. Hence proved.

Theorem 2.1.2

Let G be a finite group. Then

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{|S(G, \text{Aut}(G))|} \left(1 + \frac{|S(G, \text{Aut}(G))| - 1}{|G:L(G)|} \right)$$

The equality holds if and only if

$$\text{orb}_G(x) = xS(G, \text{Aut}(G))$$

for all

$$x \in G \setminus L(G)$$

Proof

By the orbit formula for autocommuting probability,

$$\Pr(G, \text{Aut}(G)) = \frac{1}{|G|} \sum_{x \in G} \frac{1}{|\text{orb}_G(x)|}$$

Now divide the summation into two parts:

$$G = L(G) \cup (G \setminus L(G))$$

If

$$x \in L(G)$$

then every automorphism of G fixes x . Hence,

$$\text{orb}_G(x) = \{x\}$$

and so

$$|\text{orb}_G(x)| = 1$$

Therefore, each element of $L(G)$ contributes 1 to the sum.

Now let

$$x \in G \setminus L(G)$$

For every $\alpha \in \text{Aut}(G)$, the autocommutator is

$$[x, \alpha] = x^{-1}\alpha(x)$$

Thus,

$$\alpha(x) = x[x, \alpha]$$

Since

$$[x, \alpha] \in S(G, \text{Aut}(G))$$

we get

$$\alpha(x) \in xS(G, \text{Aut}(G))$$

Hence,

$$\text{orb}_G(x) \subseteq xS(G, \text{Aut}(G))$$

Therefore,

$$|\text{orb}_G(x)| \leq |S(G, \text{Aut}(G))|.$$

Taking reciprocals gives

$$\frac{1}{|\text{orb}_G(x)|} \geq \frac{1}{|S(G, \text{Aut}(G))|}$$

Using this in the orbit formula,

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{|G|} \left(|L(G)| + \frac{|G| - |L(G)|}{|S(G, \text{Aut}(G))|} \right)$$

Let

$$m = |S(G, \text{Aut}(G))|$$

Then

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{|G|} \left(|L(G)| + \frac{|G| - |L(G)|}{m} \right)$$

Simplifying,

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{m|G|} (m|L(G)| + |G| - |L(G)|)$$

Hence,

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{m|G|} (|G| + (m-1)|L(G)|)$$

Since

$$|G:L(G)| = \frac{|G|}{|L(G)|}$$

we obtain

$$\frac{|L(G)|}{|G|} = \frac{1}{|G:L(G)|}$$

Therefore,

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{m} \left(1 + \frac{m-1}{|G:L(G)|} \right)$$

Substituting

$$m = |S(G, \text{Aut}(G))|$$

we get

$$\Pr(G, \text{Aut}(G)) \geq \frac{1}{|S(G, \text{Aut}(G))|} \left(1 + \frac{|S(G, \text{Aut}(G))| - 1}{|G:L(G)|} \right)$$

This proves the required inequality.

Now consider the equality condition. In the proof, the only inequality used was

$$|\text{orb}_G(x)| \leq |S(G, \text{Aut}(G))|$$

for

$$x \in G \setminus L(G)$$

Therefore, equality holds precisely when

$$|\text{orb}_G(x)| = |S(G, \text{Aut}(G))|$$

for every

$$x \in G \setminus L(G)$$

Since we already have

$$\text{orb}_G(x) \subseteq xS(G, \text{Aut}(G)),$$

equality of cardinalities gives

$$\text{orb}_G(x) = xS(G, \text{Aut}(G)).$$

Hence equality holds if and only if

$$\text{orb}_G(x) = xS(G, \text{Aut}(G))$$

for all

$$x \in G \setminus L(G)$$

Thus the theorem is proved.

Lemma 2.1.3

Let G be a finite group. Then, for any two integers $m \geq n$,

$$\frac{1}{n} \left(1 + \frac{n-1}{|G:L(G)|} \right) \geq \frac{1}{m} \left(1 + \frac{m-1}{|G:L(G)|} \right)$$

If $L(G) \neq G$, then equality holds if and only if

$$m = n$$

Proof

Let

$$k = |G:L(G)|.$$

Then the required inequality becomes

$$\frac{1}{n} \left(1 + \frac{n-1}{k} \right) \geq \frac{1}{m} \left(1 + \frac{m-1}{k} \right)$$

Now simplify the left-hand side:

$$\frac{1}{n} \left(1 + \frac{n-1}{k} \right) = \frac{1}{n} \left(\frac{k+n-1}{k} \right) = \frac{k+n-1}{nk}.$$

Similarly,

$$\frac{1}{m} \left(1 + \frac{m-1}{k} \right) = \frac{k+m-1}{mk}$$

Thus, it is enough to prove that

$$\frac{k+n-1}{nk} \geq \frac{k+m-1}{mk}$$

Since $k > 0$, this is equivalent to

$$\frac{k+n-1}{n} \geq \frac{k+m-1}{m}$$

Now,

$$\frac{k+n-1}{n} = 1 + \frac{k-1}{n}$$

and

$$\frac{k+m-1}{m} = 1 + \frac{k-1}{m}.$$

Since

$$m \geq n$$

we have

$$\frac{1}{n} \geq \frac{1}{m}$$

Therefore,

$$\frac{k-1}{n} \geq \frac{k-1}{m}$$

provided $k-1 \geq 0$. Hence,

$$1 + \frac{k-1}{n} \geq 1 + \frac{k-1}{m}$$

Thus,

$$\frac{1}{n} \left(1 + \frac{n-1}{|G:L(G)|} \right) \geq \frac{1}{m} \left(1 + \frac{m-1}{|G:L(G)|} \right)$$

Now suppose

$$L(G) \neq G$$

Then

$$|G:L(G)| > 1,$$

so

$$k-1 > 0.$$

Equality holds exactly when

$$\frac{k-1}{n} = \frac{k-1}{m}.$$

Since $k - 1 > 0$, this implies

$$\frac{1}{n} = \frac{1}{m}$$

Hence,

$$m = n$$

Conversely, if $m = n$, then both sides are clearly equal.

Therefore, if $L(G) \neq G$, equality holds if and only if

$$m = n$$

Hence proved.

Conclusion

In this paper, we investigated the concept of g -autocommutativity in finite groups and examined its significance as a quantitative measure of the interaction between group elements and automorphisms. By extending the classical notion of autocommuting probability, the study provided a refined probabilistic framework for analyzing the distribution of autocommutator values in finite groups.

A general counting formula for the g -autocommuting probability $\text{Pr}_g(G, \text{Aut}(G))$ was established using automorphism centralizers and orbit structures arising from the natural action of $(\text{Aut}(G))$ on (G) . This formula enabled the computation of autocommuting probabilities through structural invariants such as automorphism stabilizers and orbits. The results demonstrated that the occurrence of a prescribed autocommutator value is closely related to the orbit decomposition of the group under automorphism action.

Several upper and lower bounds for $\text{Pr}_g(G, \text{Aut}(G))$ were also derived in terms of the absolute center $(L(G))$, the subgroup $(S(G, \text{Aut}(G)))$, automorphism centralizers, and the smallest prime divisor of $(\text{Aut}(G))$. These bounds reveal how the internal symmetry of a finite group influences its autocommutativity behavior. In particular, the obtained inequalities provide useful estimates for groups whose exact autocommuting probabilities are difficult to determine.

The study further established conditions under which equality holds in the derived bounds, leading to structural characterizations involving orbit sizes and automorphism actions. These results illustrate that autocommuting probability is not merely a numerical invariant but also a powerful tool for understanding the relationship between group structure and automorphism-induced symmetries.

Overall, the investigation confirms that g -autocommuting probability serves as an effective probabilistic invariant for finite groups. The developed formulae, bounds, and characterization results contribute to the growing area of probabilistic group theory and provide a foundation for future research on generalized autocommutativity, relative autocommuting probabilities, and automorphism-based invariants in finite algebraic systems.

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