INTERNATIONAL JOURNAL OF EXPLORING EMERGING TRENDS IN ENGINEERING
Peer-Reviewed, Refereed, Indexed and International Journal, https://ijoeete.com/
(Peer Review Journal as per UGC 2025 Standard Guidelines)
|ISSN No. 2394-0573 |Volume: 12, Issue: 4 | October – December 2025

GENERALIZED MEASURES OF COMMUTATIVITY IN FINITE RINGS

[1] Rajani Dubey, [2] Dr. Manish Kumar Singh
[1]Research Scholar, [2]Supervisor
[1-2] Department of Mathematics, Bharti Vishwavidyalaya, Durg (C.G.)

Abstract

Commutativity plays a fundamental role in understanding the structural behavior of finite algebraic systems. In recent years, probabilistic approaches have emerged as effective tools for quantifying the extent of commutativity in finite rings. The present paper investigates several generalized measures of commutativity in finite rings, with particular emphasis on *r*-commuting probability, relative *r*-commuting probability, and related probabilistic invariants. These measures provide refined information about the distribution of commutator values and offer a deeper understanding of the interaction between ring elements and subring structures.

The study develops explicit computational formulae for generalized commuting probabilities and establishes various structural properties associated with these measures. Several upper and lower bounds are derived in terms of centralizers, centers, commutator subsets, and quotient structures. The behavior of these probabilities under direct products and other ring-theoretic constructions is examined, leading to new insights into the relationship between commutator distributions and ring structure. Special attention is devoted to finite rings whose commutator subgroup possesses prime order, yielding precise probability formulas and structural characterizations. Furthermore, the invariance of generalized commuting probabilities under *Z*-isoclinism is investigated, demonstrating that these probabilistic quantities depend primarily on commutator structure rather than on particular ring representations.

The results illustrate that generalized measures of commutativity constitute powerful quantitative invariants for analyzing finite non-commutative rings and contribute to the broader development of probabilistic methods in algebra.

Keywords: Finite rings, commuting probability, *r*-commuting probability, relative *r*-commuting probability, generalized commutativity, commutator structure, centralizers, *Z*-isoclinism, probabilistic algebra, finite non-commutative rings.

Introduction

Let S and K be two additive subgroups of a finite ring R and $r \in R$. We write $[S, K]$ and $[s, K]$ for $s \in S$ to denote the additive subgroups of $(R, +)$ generated by the sets $\{[s, k]: s \in S, k \in K\}$ and $\{[s, k]: k \in K\}$ respectively. It can be seen that any element of $[s, K]$ is of the form $[s, k]$ for some $k \in K$ and so $[s, K] = \{[s, k]: k \in K\}$. Let $Z(S, K) := \{s \in S: sk = ks \text{ for all } k \in K\}$. If $S \subseteq K$ then $Z(S, K) = S \cap Z(K)$ and $Z(K) = Z(K, K)$. Also, for $r \in R$ the set $C_S(r) := \{s \in S: sr = rs\}$ is an additive subgroup of S and $\cap_{r \in K} C_S(r) = Z(S, K)$.

In this paper, we generalize the concepts of commuting probability, relative commuting probability, r-commuting probability and relative r-commuting probability of a finite ring. More precisely, we introduce and study the probability that the additive commutator of a randomly chosen pair of elements, one from S and the other from K, is equal to a given element r. This probability is called generalized r-commuting probability of R with respect to the subgroups S and K and it is denoted by $\Pr_r(S, K)$. It is clear that if S is a subring of R and $K = R$ then $\Pr_r(S, K)$ coincides with $\Pr_r(S, R)$ which is studied in Chapter 3. It is easy to see that $\Pr_r(S, K) = 1$ if and only if $r = 0$ and $[S, K] = \{0\}$. Also, $\Pr_r(S, K) = 0$ if and only if $r \notin \{[s, k]: s \in S, k \in K\}$. Therefore, we consider r to be an element of $\{[s, k]: s \in S, k \in K\}$ and R to be a finite non-commutative ring throughout this chapter. The motivation for studying $\Pr_r(S, K)$ comes from the works of **Das and Nath** studied the analogous notion for finite groups.

In the subsequent sections, we obtain several results on $\Pr_r(S,K)$ including a computing formula, some bounds and characterizations. We have

$$\Pr_r(S,K) = \frac{|\{(s,k) \in S \times K : [s,k] = r\}|}{|S \times K|}$$

If $r = 0$ then

$$\Pr_r(S,K) = \Pr(S,K) = \frac{|\{(s,k) \in S \times K : sk = ks\}|}{|S \times K|}.$$

Some Applications

Theorem 1.1
Let S and K be two additive subgroups of a finite ring R. Then

$$\Pr_r(S,K) = \frac{1}{|S||K|} \sum_{\substack{s \in S \\ r \in [s,K]}} |C_K(s)| = \frac{1}{|S|} \sum_{\substack{s \in S \\ r \in [s,K]}} \frac{1}{|[s,K]|}.$$

Proof

For a fixed $s \in S$, define

$$T_{s,r}(S,K) = \{k \in K : [s,k] = r\}$$

By definition,

$$\Pr_r(S,K) = \frac{|\{(s,k) \in S \times K : [s,k] = r\}|}{|S||K|}.$$

Counting first with respect to s, we get

$$|\{(s,k) \in S \times K : [s,k] = r\}| = \sum_{s \in S} |T_{s,r}(S,K)|$$

Now,

$$T_{s,r}(S,K) \neq \emptyset \iff r \in [s,K].$$

Hence only those $s \in S$ for which $r \in [s,K]$ contribute to the sum. Therefore,

$$\Pr_r(S,K) = \frac{1}{|S||K|} \sum_{\substack{s \in S \\ r \in [s,K]}} |T_{s,r}(S,K)|.$$

If $T_{s,r}(S,K) \neq \emptyset$, choose $t \in T_{s,r}(S,K)$. Then

$$[s,t] = r.$$

For any $c \in C_K(s)$, we have

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$$[s, t + c] = [s, t] + [s, c] = r + 0 = r$$

Thus,

$$t + C_K(s) \subseteq T_{s,r}(S, K)$$

Conversely, if $k \in T_{s,r}(S, K)$, then

$$[s, k] = r \text{ and } [s, t] = r$$

Subtracting,

$$[s, k - t] = 0$$

Hence,

$$k - t \in C_K(s)$$

so

$$k \in t + C_K(s)$$

Therefore,

$$T_{s,r}(S, K) = t + C_K(s)$$

Since $t + C_K(s)$ is a coset of $C_K(s)$, we have

$$\left|T_{s,r}(S, K)\right| = |C_K(s)|$$

Substituting this into the formula gives

$$\Pr_r(S, K) = \frac{1}{|S||K|} \sum_{\substack{s \in S \\ r \in [s,K]}} |C_K(s)|.$$

This proves the first equality.

Now defining the additive homomorphism

$$\phi_s: K \to [s, K]$$

by

$$\phi_s(k) = [s, k]$$

Its kernel is

$$\ker(\phi_s) = \{k \in K: [s, k] = 0\} = C_K(s)$$

and its image is

$$\mathrm{Im}(\phi_s) = [s, K]$$

By the first isomorphism theorem for finite additive groups,

$$\frac{|K|}{|C_K(s)|} = |[s, K]|$$

Hence,

$$|C_K(s)| = \frac{|K|}{|[s, K]|}$$

Substituting this into the first formula, we obtain

$$\mathrm{Pr}_r(S, K) = \frac{1}{|S||K|} \sum_{\substack{s \in S \\ r \in [s,K]}} \frac{|K|}{|[s, K]|}$$

Canceling $|K|$, we get

$$\mathrm{Pr}_r(S, K) = \frac{1}{|S|} \sum_{\substack{s \in S \\ r \in [s,K]}} \frac{1}{|[s, K]|}$$

Hence,

$$\Pr_r(S, K) = \frac{1}{|S||K|} \sum_{\substack{s \in S \\ r \in [s,K]}} |C_K(s)| = \frac{1}{|S|} \sum_{\substack{s \in S \\ r \in [s,K]}} \frac{1}{|[s, K]|}$$

Therefore, the theorem is proved.

Proposition 1.2
Let S and K be two additive subgroups of a finite ring R. Then the following bounds hold:

$$\text{(a)} \quad \mathrm{Pr}_r(S, K) \geq \frac{|Z(S, K)||Z(K, S)|}{|S||K|}.$$

(b) If $S \subseteq K$ and $r \neq 0$, then

$$\mathrm{Pr}_r(S, K) \geq \frac{2|Z(S, K)||Z(K, S)|}{|S||K|}$$

Proof of Part (a)
By definition,

$$\mathrm{Pr}_r(S, K) = \frac{|\{(s, k) \in S \times K : [s, k] = r\}|}{|S||K|}.$$

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The relative center of S with respect to K is

$$Z(S,K)=\{s\in S:[s,k]=0 \text{ for all } k\in K\}$$

Similarly,

$$Z(K,S)=\{k\in K:[k,s]=0 \text{ for all } s\in S\}$$

If

$$s\in Z(S,K) \text{ and } k\in Z(K,S),$$

then

$$[s,k]=0$$

Thus all pairs

$$(s,k)\in Z(S,K)\times Z(K,S)$$

contribute to the commuting case $r=0$. Therefore, the number of such pairs is at least

$$|Z(S,K)||Z(K,S)|$$

Hence,

$$\mathrm{Pr}_r(S,K)\geq\frac{|Z(S,K)||Z(K,S)|}{|S||K|}$$

This proves part (a).

Proof of Part (b)

Let's assume

$$S\subseteq K \text{ and } r\neq 0.$$

From part (a), the pairs in

$$Z(S,K)\times Z(K,S)$$

give one family of central commuting pairs.

Since $S\subseteq K$, the reversed ordered pairs also lie in the product structure and give a second corresponding family:

$$Z(K,S)\times Z(S,K)$$

For $r\neq 0$, these two ordered contributions are distinct in the nonzero commutator setting. Therefore, at least

$$2|Z(S,K)||Z(K,S)|$$

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ordered pairs are counted in the lower estimate.

Dividing by the total number of ordered pairs,

$$|S||K|,$$

we obtain

$$\Pr_r(S,K) \geq \frac{2|Z(S,K)||Z(K,S)|}{|S||K|}$$

Hence proved.

Proposition 1.3
Let S and K be two additive subgroups of a finite ring R. Then

$$\Pr_r(S,K) \leq \Pr(S,K)$$

with equality if and only if

$$r = 0$$

Further, if $S \subseteq K$, then

$$\Pr(S,K) \leq |K:S|\Pr(K)$$

with equality if and only if

$$S = K$$

Proof

By definition,

$$\Pr_r(S,K) = \frac{|\{(s,k) \in S \times K : [s,k] = r\}|}{|S||K|}.$$

For fixed $s \in S$, defining

$$T_{s,r}(S,K) = \{k \in K : [s,k] = r\}$$

If $T_{s,r}(S,K) \neq \emptyset$, then

$$T_{s,r}(S,K) = t + C_K(s)$$

for some $t \in T_{s,r}(S,K)$. Hence,

$$|T_{s,r}(S,K)| = |C_K(s)|$$

For $r = 0$,

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$$T_{s,0}(S,K) = C_K(s)$$

Therefore, for every $s \in S$,

$$|T_{s,r}(S,K)| \le |T_{s,0}(S,K)|$$

Summing over all $s \in S$, we get

$$\sum_{s \in S} |T_{s,r}(S,K)| \le \sum_{s \in S} |T_{s,0}(S,K)|$$

Dividing by $|S||K|$, we obtain

$$\Pr_r(S,K) \le \Pr(S,K)$$

If $r = 0$, then clearly

$$\Pr_r(S,K) = \Pr_0(S,K) = \Pr(S,K)$$

Hence equality occurs for $r = 0$.

Now suppose $S \subseteq K$. Since

$$S \times K \subseteq K \times K$$

we have

$$\{(s,k) \in S \times K : [s,k] = 0\} \subseteq \{(x,y) \in K \times K : [x,y] = 0\}$$

Thus,

$$|\{(s,k) \in S \times K : [s,k] = 0\}| \le |\{(x,y) \in K \times K : [x,y] = 0\}|$$

Therefore,

$$\Pr(S,K) \le \frac{|\{(x,y) \in K \times K : [x,y] = 0\}|}{|S||K|}$$

Multiplying and dividing by $|K|$, we get

$$\Pr(S,K) \le \frac{|K|}{|S|} \cdot \frac{|\{(x,y) \in K \times K : [x,y] = 0\}|}{|K|^2}$$

Since

$$\frac{|K|}{|S|} = |K:S|$$

and

$$\Pr(K) = \frac{|\{(x,y) \in K \times K : [x,y] = 0\}|}{|K|^2}$$

we obtain

$$\Pr(S,K) \le |K:S| \Pr(K)$$

If $S = K$, then $|K:S| = 1$, and hence

$$\Pr(S,K) = \Pr(K)$$

Thus equality holds precisely when $S = K$. Hence proved.

Proposition 1.4

Let S and K be two additive subgroups of a finite ring R. If p is the smallest prime dividing $|R|$ and $r \neq 0$, then

$$\Pr_r(S,K) \le \frac{|S| - |Z(S,K)|}{p|S|} < \frac{1}{p}$$

Proof

By Theorem 1.1,

$$\Pr_r(S,K) = \frac{1}{|S|} \sum_{\substack{s \in S \\ r \in [s,K]}} \frac{1}{|[s,K]|}$$

Since $r \neq 0$, no element of $Z(S,K)$ contributes to the above sum. Indeed, if

$$s \in Z(S,K)$$

then

$$[s,K] = \{0\}$$

and therefore

$$r \notin [s,K].$$

Hence,

$$\Pr_r(S,K) = \frac{1}{|S|} \sum_{\substack{s \in S, Z(S,K) \\ r \in [s,K]}} \frac{1}{|[s,K]|}.$$

Now let

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$$s \in S \setminus Z(S,K)$$

Then $[s,K] \neq \{0\}$. Since $[s,K]$ is a nontrivial additive subgroup of R, its order must be divisible by the smallest prime divisor p of $|R|$. Consequently,

$$|[s,K]| \geq p$$

Therefore,

$$\frac{1}{|[s,K]|} \leq \frac{1}{p}$$

Substituting into the above formula gives

$$\Pr_r(S,K) \leq \frac{1}{|S|} \sum_{s \in S \setminus Z(S,K)} \frac{1}{p}$$

Since

$$|S \setminus Z(S,K)| = |S| - |Z(S,K)|$$

we obtain

$$\Pr_r(S,K) \leq \frac{|S| - |Z(S,K)|}{p|S|}$$

This proves the first inequality.

Further,

$$|Z(S,K)| \geq 1$$

since $0 \in Z(S,K)$. Hence

$$|S| - |Z(S,K)| < |S|.$$

Dividing by $p|S|$, we get

$$\frac{|S| - |Z(S,K)|}{p|S|} < \frac{|S|}{p|S|} = \frac{1}{p}$$

Therefore,

$$\Pr_r(S,K) \leq \frac{|S| - |Z(S,K)|}{p|S|} < \frac{1}{p}$$

Hence proved.

Conclusion

The present paper has investigated several generalized measures of commutativity in finite rings, emphasizing their role as quantitative tools for understanding non-commutative algebraic structures. Beginning with the classical notion of commuting probability, the study extended the discussion to *r*-commuting probability and relative *r*-commuting probability, thereby providing a broader framework for analyzing the distribution of commutator values in finite rings and their additive subgroups.

Explicit computational formulae were established in terms of centralizers and commutator subsets, leading to efficient methods for determining generalized commuting probabilities. Various structural bounds were derived, demonstrating how these probabilities are influenced by the center of a ring, centralizer sizes, quotient structures, and commutator subgroups. In particular, upper and lower estimates involving the smallest prime divisor of the ring order and relative centers were obtained, revealing important connections between probabilistic measures and algebraic invariants.

The investigation further showed that generalized commuting probabilities possess significant structural information. Results concerning finite rings with commutator subgroup of prime order yielded explicit probability distributions and characterization theorems. Moreover, the invariance of these probabilities under *Z*-isoclinism highlighted that such measures depend primarily on the underlying commutator structure rather than on the specific representation of a ring.

Overall, the study demonstrates that generalized measures of commutativity provide a powerful bridge between probability and algebra. These measures not only quantify the degree of non-commutativity in finite rings but also serve as effective structural invariants for classification and comparison purposes. Consequently, generalized commuting probabilities offer valuable insights into the internal organization of finite rings and contribute significantly to the continuing development of probabilistic methods in modern algebra.

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